Lecture Continuum Mechanic
SeaTech 1st year

Part 1

Tensorial Notations

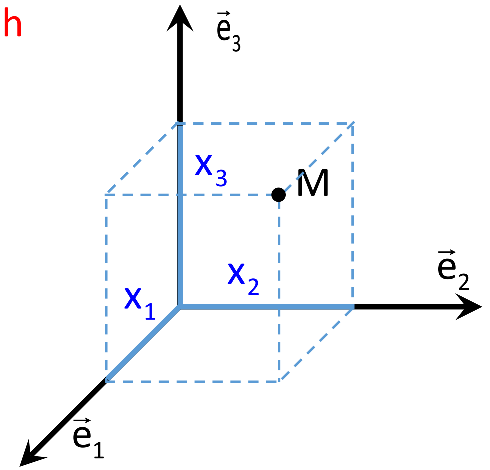
F. Golay

Vectors and tensors

Particular case usually met, which simplify the notations

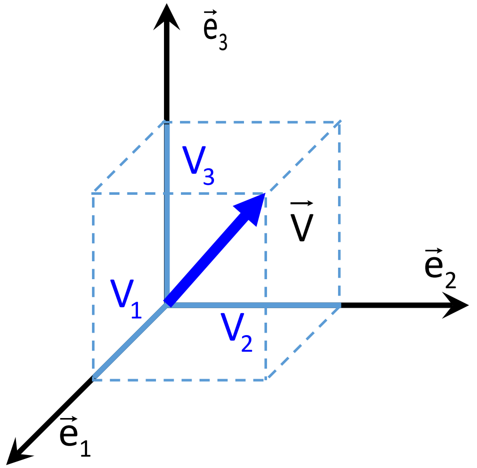
Vector

We consider an euclidian space $\mathcal{E}$ with dimension 3
and an orthonormal basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$
The position of any point M, at time t, is described by its coordinates x_1, x_2, x_3.



Let $\vec{V}$ be a time dependent vector field

$$\vec{V}(\vec{x},t) = \{\vec{V}\} = \begin{Bmatrix} V_1(x_1,x_2,x_3,t) \\ V_2(x_1,x_2,x_3,t) \\ V_3(x_1,x_2,x_3,t) \end{Bmatrix} = V_1\vec{e}_1 + V_2\vec{e}_2 + V_3\vec{e}_3 = \sum_{i=1}^{3} V_i\vec{e}_i$$



Einstein's summation convention

$$\vec{V} = \sum_{i=1}^{3} V_i\vec{e}_i = V_i\vec{e}_i$$

i « dummy » index

Rmk 1: $\vec{V} = V_i\vec{e}_i = V_k\vec{e}_k$

Rmk 2: transpose vector $\vec{V}^T = \{\vec{V}\}^T = \langle\vec{V}\rangle = \langle V_1, V_2, V_3\rangle$

Tensorial Notations

Vectors and tensors

Permutations and determinants

Vectorial Analysis

Cylindrical coordonates

Vectors and tensors

Linear application from $\mathcal{E}$ to $\mathcal{E}$

Let A be a linear application from $\mathcal{E}$ to $\mathcal{E}$ represented by a 3x3 matrix [A]

$$[A]=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Let us consider $\vec{W}$ the image of a vector $\vec{V}$ by the linear application A

$$\vec{W}=A\vec{V} \quad ; \quad \{\vec{W}\}=[A]\{\vec{V}\} \quad ; \quad \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \end{Bmatrix}=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix}=\begin{Bmatrix} A_{11}V_1+A_{12}V_2+A_{13}V_3 \\ A_{21}V_1+A_{22}V_2+A_{23}V_3 \\ A_{31}V_1+A_{32}V_2+A_{33}V_3 \end{Bmatrix}$$

Einstein's summation convention

$$W_i\vec{e}_i=A_{ij}V_j\vec{e}_i$$

Particular case of the identity application, which can be represented using the Kronecker symbols

$$\delta_{ij}=\begin{cases} 1 & \text{si} \quad i=j \\ 0 & \text{si} \quad i\neq j \end{cases} \qquad [I]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix}$$

Vectors and tensors

Bilinear application from $\mathcal{E}x\mathcal{E}$ to $\mathbb{R}$

Let A be a bilinear application from $\mathcal{E}$ x $\mathcal{E}$ to $\mathbb{R}$ represented by the matrix A such that

$$A(\vec{V},\vec{W}) = \langle \vec{V} \rangle [A] \{\vec{W}\} = \langle V_1 \quad V_2 \quad V_3 \rangle \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \end{Bmatrix} = A_{ij} V_i W_j$$

Particular case of the scalar product

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

$$\vec{V} \cdot \vec{W} = \langle \vec{V} \rangle \{\vec{W}\} = (V_i \vec{e}_i) \cdot (W_j \vec{e}_j) = V_i W_j \vec{e}_i \cdot \vec{e}_j = V_i W_j \delta_{ij} = V_i W_i$$

Tensorial Notations

Vectors and tensors

Permutations and determinants

Vectorial Analysis

Cylindrical coordonates

Vectors et tensors

Tensor

Wikipedia Definition:

Let V be a n-dimensional vectorial space on commutative field K; the dual space V is the vectorial space composed of all linear form defined on V. V* is also a n-dimensional space. The elements of V and V* are called respectively vector and co-vector. A tensor of order (h+k) is a multilinear application* $T:\underbrace{V^*\times\cdots\times V^*}_{h}\times\underbrace{V\times\cdots\times V}_{k}\to K$

Lecture Samuel Forest : http://mms2.ensmp.fr/mmc_paris/amphis/tenseurs.pdf

Lecture Jean Garrigues : http://jean.garrigues.perso.centrale-marseille.fr/tenseurs.html

Jean Salençon : « Mécanique des Milieux Continus », tome 1, ed. Ecole polytechnique, 2005.

Recall: We consider only a 3-dimensional euclidian space $\mathcal{E}$ with ortogonal basis

Jean Coirier : « Mécanique des Milieux Continus », ed. Dunod, Paris, 2001 (at SeaTech library!).

Definition : *We call Tensor of order n on* $\mathcal{E}$ *every n-linear form* $T:\underbrace{\mathcal{E}\times\cdots\times\mathcal{E}}_{\text{n times}}$

So let's take a simplifying shortcut….

A tenseur of order 0 « is » a scalar with 3^0=1 component
A tensor of order 1 « is » a vector with 3^1=3 components
A tensor of order 2 « is » a matrix with 3^2=9 components
.. etc …

Tensorial Notations

Vectors and tensors

Second order tensor

A second order tensor T is a linear operator which, to every vector $\vec{V}$ from the euclidian space, associates a vector $\vec{W}$ from the same space

$$\vec{W} = T\left(\vec{V}\right) \qquad W_i = T_{ij}V_j \qquad \text{Notation}: T, [T], \bar{\bar{T}}$$

- A tensor is said symmetric if $T_{ij} = T_{ji}$
- A tensor is said antisymmetric if $T_{ij} = -T_{ji}$
- A tensor is said isotropic if $T_{ij} = t\delta_{ij}$

Rmk: we can always split a tensor with a symmetric part and antisymmetric part

$$T_{ij} = \underbrace{\frac{1}{2}\left(T_{ij} + T_{ji}\right)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}\left(T_{ij} - T_{ji}\right)}_{\text{Antiymmetric}}$$

Tensorial Notations

Vectors and tensors

Permutations and determinants

Vectorial Analysis

Cylindrical coordonates

Vectors and tensors

Tensorial product

We define the tensorial product of the vector $\vec{U}$ by the vector $\vec{V}$, noted $\vec{U} \otimes \vec{V}$, as the second order tensor, defined by the bilinear form that associates to the vectors $\vec{X}$ and $\vec{Y}$ the quantity $(\vec{U} \cdot \vec{X})(\vec{V} \cdot \vec{Y})$

$$\vec{u} \otimes \vec{v} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

The 9 tensorial products $\vec{e}_i \otimes \vec{e}_j$ define a basis of the second order tensor space

$$\vec{e}_1 \otimes \vec{e}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{e}_1 \otimes \vec{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{… etc …}$$

$$\overline{\overline{T}} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = T_{ij}\vec{e}_i \otimes \vec{e}_j \qquad \vec{u} \otimes \vec{v} = u_i v_j \vec{e}_i \otimes \vec{e}_j$$

Vectors and tensors

Contracted tensor product

The contraction of 2 first order tensors

$$\vec{u}\cdot\vec{v}=(u_i\vec{e}_i)\cdot(v_j\vec{e}_j)=u_iv_j\vec{e}_i\cdot\vec{e}_j=u_iv_j\delta_{ij}=u_iv_i=u_1v_1+u_2v_2+u_3v_3$$

It is the scalar product of 2 vectors !

The contraction of a second order tensor and a first order tensor

$$\overline{\overline{T}}\cdot\vec{v}=(T_{ij}\vec{e}_i\otimes\vec{e}_j)\cdot(v_k\vec{e}_k)=T_{ij}v_k\vec{e}_i\otimes\vec{e}_j\cdot\vec{e}_k=T_{ik}v_k\vec{e}_i$$

It is the product of a matrix and a vector !

P-order tensor « contracted » with q-order tensor = (p+q-2)-order tensor

Double contraction of 2 second order tensors

$$\overline{\overline{A}}:\overline{\overline{B}}=(A_{ij}\vec{e}_i\otimes\vec{e}_j):(B_{pq}\vec{e}_p\otimes\vec{e}_q)=A_{ij}B_{pq}\vec{e}_i\otimes\vec{e}_j:\vec{e}_p\otimes\vec{e}_q=A_{ip}B_{pq}\vec{e}_i\cdot\vec{e}_q=A_{ip}B_{pi}$$

Double contraction of a second order tensor and the **identity** tensor

$$\overline{\overline{A}}:\overline{\overline{I}}=A_{ip}\delta_{pi}=A_{pp}=A_{11}+A_{22}+A_{33}=Tr\left(\overline{\overline{A}}\right)$$

Permutation and determinant

Permutation symbols

Tensorial Notations

Vectors and tensors

Permutations and determinants

Vectorial Analysis

Cylindrical coordonates

We define the permutation symbols or Levi-Civita 3-order tensor by

$$\varepsilon_{ijk} = \left(\vec{e}_i, \vec{e}_j, \vec{e}_k\right) = \begin{cases} +1 & \text{if i, j,k is a permutation of 1,2,3 i.e. : 123, 231, 312} \\ -1 & \text{if i, j,k is a permutation de 2,1,3 i.e. : 213, 132, 321} \\ 0 & \text{if two indices are repeated} \end{cases}$$

Triple product

We can prove the folowing results :

$$\begin{cases} \varepsilon_{ijk}\varepsilon_{lmn} = \mathrm{Det}\begin{bmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{bmatrix} \\ \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \\ \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn} \\ \varepsilon_{ijk}\varepsilon_{ijk} = 6 \end{cases}$$

Permutation and determinants

Determinant

$$\mathrm{Det}(A) = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{mnp}A_{im}A_{jn}A_{kp}$$

Cross product

$$\vec{c} = \vec{a}\times\vec{b}$$

$$c_i\vec{e}_i = \varepsilon_{ijk}a_j b_k \vec{e}_i$$

Example : Prove that , $\left(\vec{a}\times\vec{b}\right)\times\vec{c} = \left(\vec{a}\cdot\vec{c}\right)\vec{b} - \left(\vec{b}\cdot\vec{c}\right)\vec{a}$

Vectorial analysis

Notation

Important: We consider a 3-dimensional euclidian space $\mathcal{E}$ with orthonormal basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$.
Each tensor field depends of the variables t, x_1, x_2, x_3.

Partial derivative notation $_{,i} \leftrightarrow \frac{\partial}{\partial x_i}$

Gradient operator : $\nabla(*) = (*)_{,j} \otimes \vec{e}_j$ Order +1

Divergence operator : $\mathrm{div}(*) = \nabla(*) : \bar{\bar{I}} = (*)_{,j} \cdot \vec{e}_j$ Order -1

Tensorial Notations

Vectors and tensors

Permutations and determinants

Vectorial Analysis

Cylindrical coordonates

Tensorial Notations

Vectorial analysis

Let f be a scalar field

The gradient of a scalar field is a vector field

$$\overrightarrow{\text{grad}}\, f = \nabla f = \begin{Bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial f}{\partial x_3} \end{Bmatrix} = f_{,i}\, \vec{e}_i$$

The Laplacian of a scalar field is a scalar field

$$\Delta f = \text{div}(\nabla f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = f_{,ii}$$

Vectorial analysis

Let $\vec{V}$ be a vector field

The gradient of a vector field is a matrix field

$$\nabla \vec{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} = \left(v_i\, \vec{e}_i\right)_{,j} \otimes \vec{e}_j = v_{i,j}\, \vec{e}_i \otimes \vec{e}_j$$

The Laplacien of a vector field is a vector field

$$\Delta \vec{v} = \begin{Bmatrix} \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2} \end{Bmatrix} = \begin{Bmatrix} \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \end{Bmatrix} = \operatorname{div}\left(\nabla \vec{v}\right) = \left(v_{i,j}\, \vec{e}_i \otimes \vec{e}_j\right)_{,k} \cdot \vec{e}_k = v_{i,jj}\, \vec{e}_i$$

Vectorial analysis

Let $\vec{V}$ be a vector field

The divergence of a vector field is a scalar field

$$\mathrm{Div}\,\vec{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \left(v_i\,\vec{e}_i\right)_{,j} \cdot \vec{e}_j = v_{i,j}\,\vec{e}_i \cdot \vec{e}_j = v_{i,i}$$

The curl of a vector field is a vector field

$$\mathrm{curl}\,\vec{v} = \nabla \times \vec{v} = \begin{Bmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\ \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{Bmatrix} = \varepsilon_{ijk}\, v_{k,j}\, \vec{e}_i$$

Vectorial analysis

Let $\overline{\overline{T}}$ be a second order tensor field

The divergence of a second order tensor field is a vector field

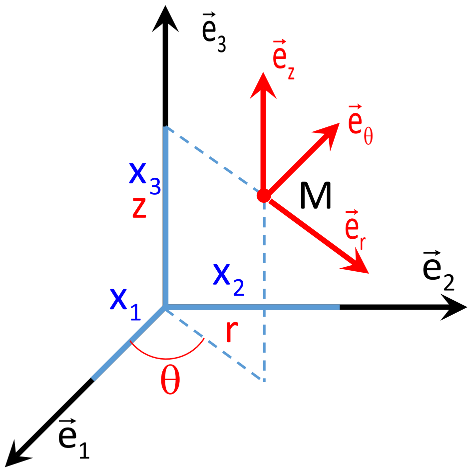
$$\operatorname{Div}\overline{\overline{T}} = \begin{Bmatrix} \dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial T_{12}}{\partial x_2} + \dfrac{\partial T_{13}}{\partial x_3} \\ \dfrac{\partial T_{21}}{\partial x_1} + \dfrac{\partial T_{22}}{\partial x_2} + \dfrac{\partial T_{23}}{\partial x_3} \\ \dfrac{\partial T_{31}}{\partial x_1} + \dfrac{\partial T_{32}}{\partial x_2} + \dfrac{\partial T_{33}}{\partial x_3} \end{Bmatrix} = \left(T_{ij}\,\vec{e}_i \otimes \vec{e}_j\right)_{,k} \cdot \vec{e}_k = T_{ij,j}\,\vec{e}_i$$

Example : Prove that , $\operatorname{Div}\left(\vec{a}\times\vec{b}\right) = \vec{b}\cdot\operatorname{curl}\vec{a} - \vec{a}\cdot\operatorname{curl}\vec{b}$

Cylindrical coordinates

Cylindrical coordinates

$$\overrightarrow{OM} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 = r\cos\theta\vec{e}_1 + r\sin\theta\vec{e}_2 + z\vec{e}_z = r\vec{e}_r + z\vec{e}_z$$



$$d(\overrightarrow{OM}) = \vec{e}_r dr + r d\theta \vec{e}_\theta + \vec{e}_z dz \qquad \frac{\partial\overrightarrow{OM}}{\partial r} = \vec{e}_r , \quad \frac{1}{r}\frac{\partial\overrightarrow{OM}}{\partial\theta} = \vec{e}_\theta , \quad \frac{\partial\overrightarrow{OM}}{\partial z} = \vec{e}_z$$

$$\begin{cases} \vec{e}_r = \cos\theta\vec{e}_1 + \sin\theta\vec{e}_2 \\ \vec{e}_\theta = -\sin\theta\vec{e}_1 + \cos\theta\vec{e}_2 \\ \vec{e}_z = \vec{e}_3 \end{cases}$$

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \\ x_3 = z \end{cases}$$

$$\frac{\partial\vec{e}_r}{\partial r} = 0 , \quad \frac{\partial\vec{e}_\theta}{\partial r} = 0 , \quad \frac{\partial\vec{e}_z}{\partial r} = 0$$

$$\frac{\partial\vec{e}_r}{\partial\theta} = \vec{e}_\theta , \quad \frac{\partial\vec{e}_\theta}{\partial\theta} = -\vec{e}_r , \quad \frac{\partial\vec{e}_z}{\partial\theta} = 0$$

$$\frac{\partial\vec{e}_r}{\partial z} = 0 , \quad \frac{\partial\vec{e}_\theta}{\partial z} = 0 , \quad \frac{\partial\vec{e}_z}{\partial z} = 0$$

Notation: for i=r,θ,z we consider $\quad ,i = \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}$ $\qquad \vec{e}_{r,\theta} = \frac{\vec{e}_\theta}{r}$ et $\vec{e}_{\theta,\theta} = -\frac{\vec{e}_r}{r}$

Tensorial Notations

Vectors and tensors

Permutations and determinants

Vectorial Analysis

Cylindrical coordonates

Cylindrical coordinates

Formulas using cylindrical coordinates

Let f(r,θ,z) be a scalar field, then

$$\overrightarrow{\text{grad}}(f) = \nabla f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\begin{aligned}\nabla f &= \left(f\right)_{,i} \otimes \vec{e}_i \\ &= f_{,i}\vec{e}_i \\ &= \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{\partial f}{\partial z}\vec{e}_z\end{aligned}$$

$$\begin{aligned}\Delta f &= \text{div}\left(\nabla f\right) \\ &= \left(f_{,i}\vec{e}_i\right)_{,j} \cdot \vec{e}_j \\ &= f_{,ij}\vec{e}_i \cdot \vec{e}_j + f_{,i}\vec{e}_{i,j} \cdot \vec{e}_j \\ &= f_{,ii} + f_{,i}\vec{e}_{i,\theta} \cdot \vec{e}_\theta \\ &= f_{,ii} + f_{,r}\frac{\vec{e}_\theta}{r} \cdot \vec{e}_\theta - f_{,\theta}\frac{\vec{e}_r}{r} \cdot \vec{e}_\theta \\ &= f_{,ii} + \frac{f_{,r}}{r} \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned}$$

Tensorial Notations

Vectors and tensors

Permutations and determinants

Vectorial Analysis

Cylindrical coordonates

Cylindrical coordinates

Formulas using cylindrical coordinates

Let $\vec{V}(r,\theta,z)$ be a vector field, then

$$\mathrm{grad}(\vec{v}) = \overline{\overline{\nabla v}} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

$$\begin{aligned} \nabla(\vec{V}) &= (V_i \vec{e}_i)_{,j} \otimes \vec{e}_j \\ &= V_{i,j}\vec{e}_i \otimes \vec{e}_j + V_i \vec{e}_{i,j} \otimes \vec{e}_j \\ &= V_{i,j}\vec{e}_i \otimes \vec{e}_j + V_i \vec{e}_{i,\theta} \otimes \vec{e}_\theta \\ &= V_{i,j}\vec{e}_i \otimes \vec{e}_j + V_r \vec{e}_{r,\theta} \otimes \vec{e}_\theta + V_\theta \vec{e}_{\theta,\theta} \otimes \vec{e}_\theta \\ &= V_{i,j}\vec{e}_i \otimes \vec{e}_j + \frac{V_r}{r}\vec{e}_\theta \otimes \vec{e}_\theta - \frac{V_\theta}{r}\vec{e}_r \otimes \vec{e}_\theta \end{aligned}$$

$$\mathrm{div}\,\vec{v} = \mathrm{Tr}\left(\nabla(\vec{v})\right) = \nabla\vec{v} : \overline{\overline{I}} = \frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

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Tensorial Notations

Cylindrical coordinates

Formulas using cylindrical coordinates

Let $\vec{V}(r,\theta,z)$ be a vector field, then

$$\Delta\vec{V} = \mathrm{div}\left(\nabla\vec{V}\right) = \left(\Delta V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta} - \frac{V_r}{r^2}\right)\vec{e}_r + \left(\Delta V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r^2}\right)\vec{e}_\theta + \Delta V_z\vec{e}_z$$

$$\mathrm{div}\left(\nabla\vec{V}\right) = \left(V_{i,j}\vec{e}_i\otimes\vec{e}_j + \frac{V_r}{r}\vec{e}_\theta\otimes\vec{e}_\theta - \frac{V_\theta}{r}\vec{e}_r\otimes\vec{e}_\theta\right)_{,k}\cdot\vec{e}_k$$

$$= V_{i,jk}\vec{e}_i\otimes\vec{e}_j\cdot\vec{e}_k + V_{i,j}\vec{e}_{i,\theta}\otimes\vec{e}_j\cdot\vec{e}_\theta + V_{i,j}\vec{e}_i\otimes\vec{e}_{j,\theta}\cdot\vec{e}_\theta + \left(\frac{V_r}{r}\right)_{,k}\vec{e}_\theta\otimes\vec{e}_\theta\cdot\vec{e}_k - \left(\frac{V_\theta}{r}\right)_{,k}\vec{e}_r\otimes\vec{e}_\theta\cdot\vec{e}_k$$

$$+\frac{V_r}{r}\vec{e}_{\theta,\theta}\otimes\vec{e}_\theta\cdot\vec{e}_\theta + \frac{V_r}{r}\vec{e}_\theta\otimes\vec{e}_{\theta,\theta}\cdot\vec{e}_\theta - \frac{V_\theta}{r}\vec{e}_{r,\theta}\otimes\vec{e}_\theta\cdot\vec{e}_\theta - \frac{V_\theta}{r}\vec{e}_r\otimes\vec{e}_{\theta,\theta}\cdot\vec{e}_\theta$$

$$= V_{i,kk}\vec{e}_i + V_{i,\theta}\vec{e}_{i,\theta} + V_{i,r}\vec{e}_i\otimes\vec{e}_{r,\theta}\cdot\vec{e}_\theta + \left(\frac{V_r}{r}\right)_{,\theta}\vec{e}_\theta - \left(\frac{V_\theta}{r}\right)_{,\theta}\vec{e}_r + \frac{V_r}{r}\vec{e}_{\theta,\theta} - \frac{V_\theta}{r}\vec{e}_{r,\theta}$$

$$= V_{i,kk}\vec{e}_i + \frac{1}{r}V_{r,\theta}\vec{e}_\theta - \frac{1}{r}V_{\theta,\theta}\vec{e}_r + \frac{1}{r}V_{i,r}\vec{e}_i + \frac{V_{r,\theta}}{r}\vec{e}_\theta - \frac{V_{\theta,\theta}}{r}\vec{e}_r - \frac{V_r}{r^2}\vec{e}_r - \frac{V_\theta}{r^2}\vec{e}_\theta$$

$$= \left(\Delta V_i\right)\vec{e}_i + \left(-2\frac{V_{\theta,\theta}}{r} - \frac{V_r}{r^2}\right)\vec{e}_r + \left(2\frac{V_{r,\theta}}{r} - \frac{V_\theta}{r^2}\right)\vec{e}_\theta$$

Cylindrical coordinates

Formulas using cylindrical coordinates

Let $\vec{V}(r,\theta,z)$ be a vector field, then

$$\text{rot}\vec{V} = \nabla\times\vec{V} = \left(\frac{1}{r}\frac{\partial V_z}{\partial\theta} - \frac{\partial V_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\vec{e}_z$$

$$\text{curl}\vec{V} = \overline{\overline{\overline{\varepsilon}}} : \nabla\vec{V}$$

$$= \left(\varepsilon_{mnp}\vec{e}_m\otimes\vec{e}_n\otimes\vec{e}_p\right) : \left(V_{i,j}\vec{e}_i\otimes\vec{e}_j + \frac{V_r}{r}\vec{e}_\theta\otimes\vec{e}_\theta - \frac{V_\theta}{r}\vec{e}_r\otimes\vec{e}_\theta\right)$$

$$= \varepsilon_{mnp}V_{i,j}\vec{e}_m\otimes\vec{e}_n\otimes\vec{e}_p : \vec{e}_i\otimes\vec{e}_j + \varepsilon_{mnp}\frac{V_r}{r}\vec{e}_m\otimes\vec{e}_n\otimes\vec{e}_p : \vec{e}_\theta\otimes\vec{e}_\theta - \varepsilon_{mnp}\frac{V_\theta}{r}\vec{e}_m\otimes\vec{e}_n\otimes\vec{e}_p : \vec{e}_r\otimes\vec{e}_\theta$$

$$= \varepsilon_{mnp}V_{p,n}\vec{e}_m + \varepsilon_{m\theta\theta}\frac{V_r}{r}\vec{e}_m - \varepsilon_{m\theta r}\frac{V_\theta}{r}\vec{e}_m$$

$$= \varepsilon_{mnp}V_{p,n}\vec{e}_m - \varepsilon_{z\theta r}\frac{V_\theta}{r}\vec{e}_z$$

$$= \varepsilon_{mnp}V_{p,n}\vec{e}_m - \frac{V_\theta}{r}\vec{e}_z$$

Tensorial Notations

Vectors and tensors

Permutations and determinants

Vectorial Analysis

Cylindrical coordonates

Cylindrical coordinates

Formulas using cylindrical coordinates

Let $\overline{\overline{T}}(r,\theta,z)$ be a 2-order tensor field, then

$$\overline{\overline{T}} = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & T_{zz} \end{bmatrix}$$

$$\mathrm{div}(\overline{\overline{T}}) = \begin{Bmatrix} \dfrac{\partial T_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{r\theta}}{\partial \theta} + \dfrac{\partial T_{rz}}{\partial z} + \dfrac{T_{rr} - T_{\theta\theta}}{r} \\ \dfrac{\partial T_{\theta r}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{\theta\theta}}{\partial \theta} + \dfrac{\partial T_{\theta z}}{\partial z} + \dfrac{T_{r\theta}}{r} + \dfrac{T_{\theta r}}{r} \\ \dfrac{\partial T_{zr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{z\theta}}{\partial \theta} + \dfrac{\partial T_{zz}}{\partial z} + \dfrac{T_{zr}}{r} \end{Bmatrix}$$

$$\begin{aligned}
\mathrm{div}(\overline{\overline{T}}) &= \left(T_{ij}\vec{e}_i \otimes \vec{e}_j\right)_{,k} \cdot \vec{e}_k \\
&= T_{ij,k}\,\vec{e}_i \otimes \vec{e}_j \cdot \vec{e}_k + T_{ij}\,\vec{e}_{i,k} \otimes \vec{e}_j \cdot \vec{e}_k + T_{ij}\,\vec{e}_i \otimes \vec{e}_{j,k} \cdot \vec{e}_k \\
&= T_{ij,k}\,\vec{e}_i \otimes \vec{e}_j \cdot \vec{e}_k + T_{ij}\,\vec{e}_{i,\theta} \otimes \vec{e}_j \cdot \vec{e}_\theta + T_{ij}\,\vec{e}_i \otimes \vec{e}_{j,\theta} \cdot \vec{e}_\theta \\
&= T_{ij,k}\,\vec{e}_i \otimes \vec{e}_j \cdot \vec{e}_k + \frac{T_{rj}}{r}\vec{e}_\theta \otimes \vec{e}_j \cdot \vec{e}_\theta - \frac{T_{\theta j}}{r}\vec{e}_r \otimes \vec{e}_j \cdot \vec{e}_\theta + \frac{T_{ir}}{r}\vec{e}_i \otimes \vec{e}_\theta \cdot \vec{e}_\theta - \frac{T_{i\theta}}{r}\vec{e}_i \otimes \vec{e}_r \cdot \vec{e}_\theta \\
&= T_{ik,k}\,\vec{e}_i + \frac{T_{r\theta}}{r}\vec{e}_\theta - \frac{T_{\theta\theta}}{r}\vec{e}_r + \frac{T_{ir}}{r}\vec{e}_i
\end{aligned}$$

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

Lecture on Continuum Mechanics
SeaTech 1st year

Part 2

Kinematics

F. Golay

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

Notion of continuous medium

Fluid: « which is neither solid nor thick, which flows easily »
Solid: « which has consistency, which is not liquid, while being more or less soft »
Liquid: "any body that flows or tends to flow“

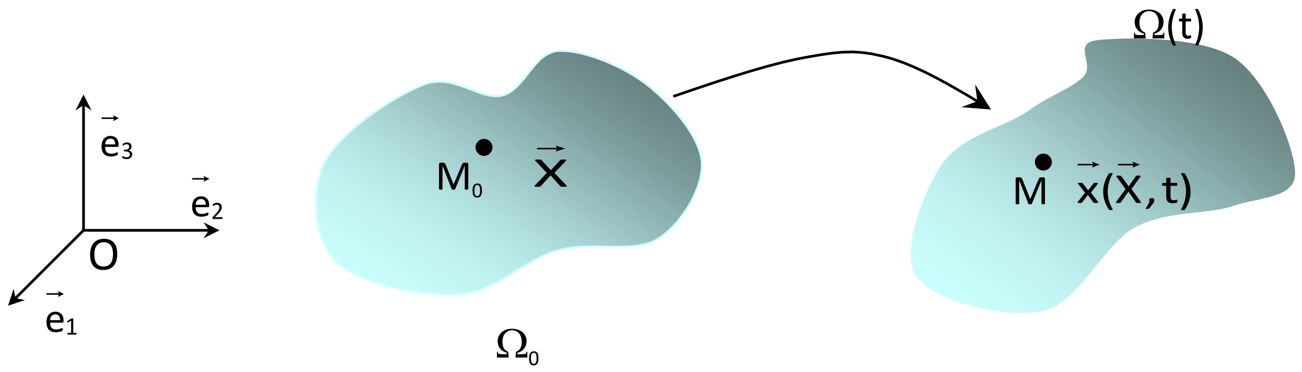
Petit Robert, french dictionary

Continuous medium: "medium whose macroscopic behavior can be schematized by assuming the matter distributed over the entire domain it occupies“

J. Coirier

The movement and its representations

Configuration



Reference configuration at time t_0

Current configuration at time t

$(\vec{X},t)$: Lagrange variables (generally used in solid mechanics)

$(\vec{x},t)$: Euler variables (generally used in fluid mechanics)

Velocity $$\vec{V}=\frac{d\overrightarrow{OM}}{dt}=\begin{Bmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\\ \frac{dx_3}{dt}\end{Bmatrix}$$

Acceleration $$\vec{\Gamma}=\frac{d\vec{V}}{dt}=\begin{Bmatrix}\frac{dV_1}{dt}\\ \frac{dV_2}{dt}\\ \frac{dV_3}{dt}\end{Bmatrix}$$

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

The movement and its representations

Trajectory

We call the trajectory of a particle the geometric curve of the positions occupied by that particle over time

$$\vec{x}(\vec{X},t)$$

In Eulerian description, knowing that $\vec{V}(\vec{x},t)=\frac{d\overrightarrow{OM}}{dt}=\frac{dx_1}{dt}\vec{e}_1+\frac{dx_2}{dt}\vec{e}_2+\frac{dx_3}{dt}\vec{e}_3$

We find the trajectory by integrating the 3 equations $\frac{dx_1}{V_1(x_1,x_2,x_3,t)}=\frac{dx_2}{V_2(x_1,x_2,x_3,t)}=\frac{dx_3}{V_3(x_1,x_2,x_3,t)}=dt$

Streamlines

At a fixed time, streamlines are a family of curves that are tangent to the velocity vector field of the flow. They show the direction in which a massless fluid element will travel at any point in time.

$$d\vec{x}\wedge\vec{V}=\vec{0}$$

At fixed t, we integrate the 2 equations $\frac{dx_1}{V_1}=\frac{dx_2}{V_2}=\frac{dx_3}{V_3}$

Rmk: for a permanent (or stationary) movement, streamlines and trajectories coincide.

$$\frac{\partial}{\partial t}=0 \qquad \vec{V}(\vec{x},t)=\vec{V}(\vec{x})$$

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

The movement and its representations

Time derivative using Lagrange variables

$$A = A(\vec{X}, t)$$

$$\frac{dA}{dt}(\vec{X}, t) = \frac{\partial A}{\partial t}(\vec{X}, t)$$

Time derivative using Euler variables

$$A = A(\vec{x}, t)$$

$$\frac{dA}{dt}\left(\vec{x}(\vec{X}, t), t\right) = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial A}{\partial t} + V_i \frac{\partial A}{\partial x_i} = \frac{\partial A}{\partial t} + \left(\vec{V} \cdot \nabla\right) A = \frac{\partial A}{\partial t} + \nabla A \cdot \vec{V}$$

Material derivative

Beware of notation abuse

Rmk: application to acceleration

$$\vec{\Gamma}(\vec{x}, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla\right)\vec{V} = \frac{\partial \vec{V}}{\partial t} + \frac{1}{2}\nabla \vec{V}^2 + \vec{rot}\,\vec{V} \wedge \vec{V}$$

Kinematics

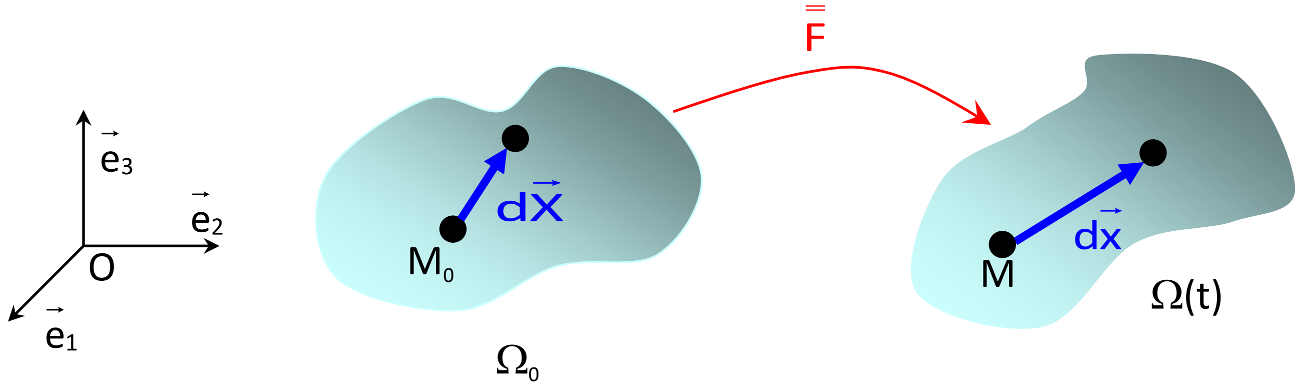
The movement and its representations

Deformation of continuous medium

Transport, material derivative

Deformation of continuous medium

Deformation gradient



Deformation gradient

$$\overline{\overline{F}} = \overline{\overline{\frac{\partial x}{\partial X}}} = \frac{\partial x_i}{\partial X_j} \vec{e}_i \otimes \vec{e}_j = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

Kinematics

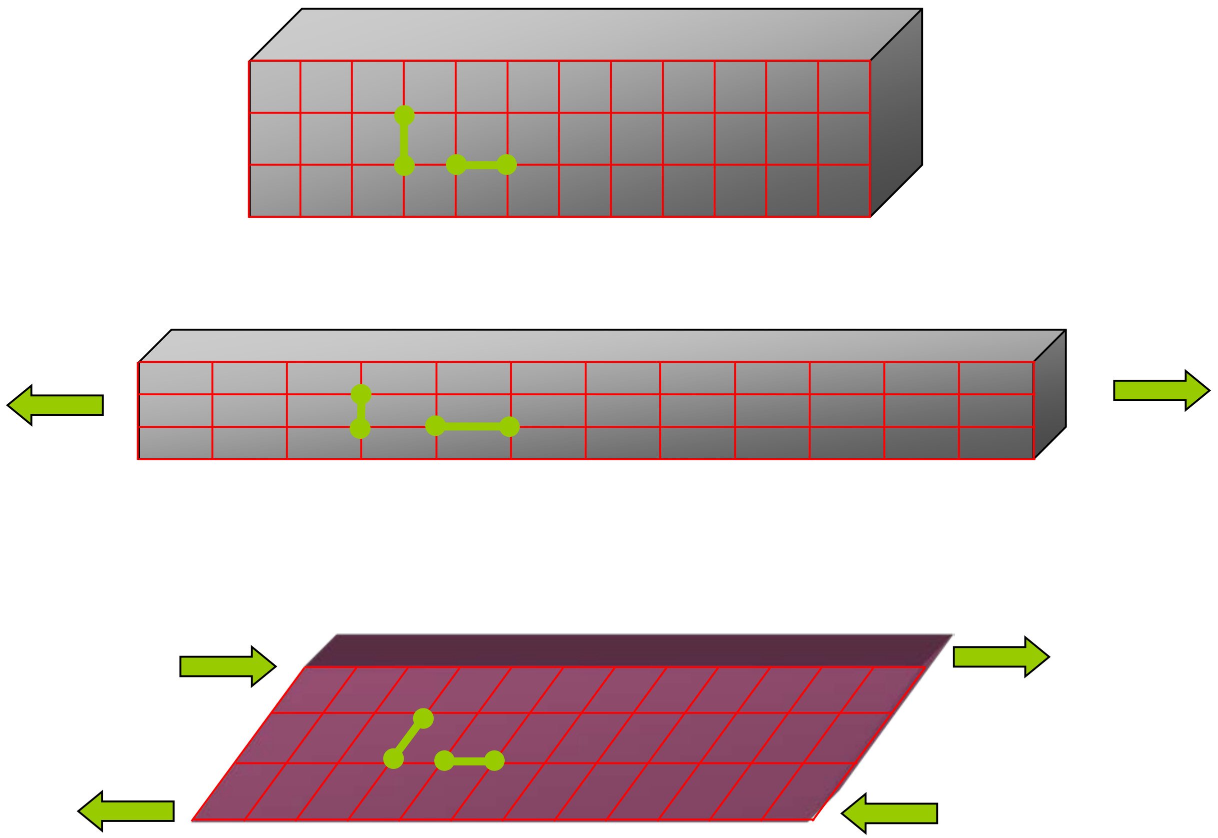
The movement and its representations

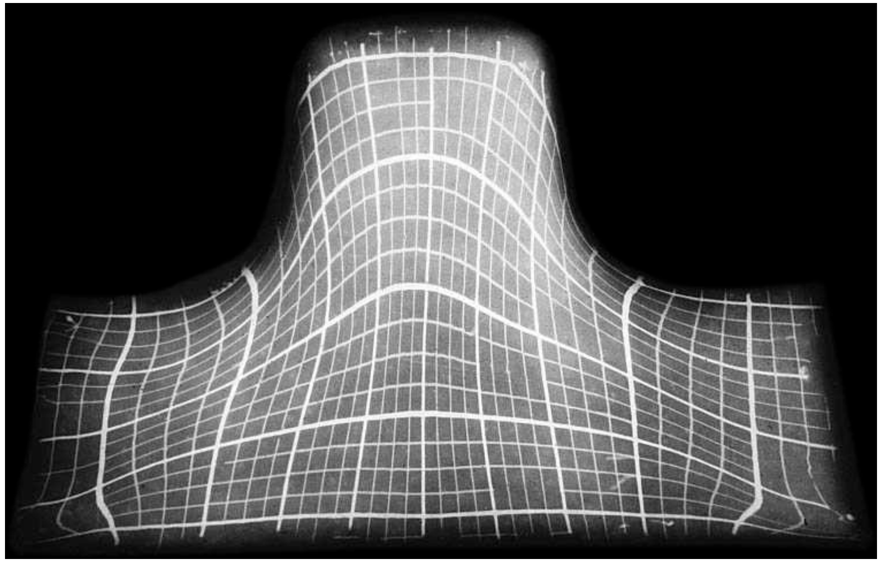
Deformation of continuous medium

Transport, material derivative

Deformation of continuous medium

Notion of deformation

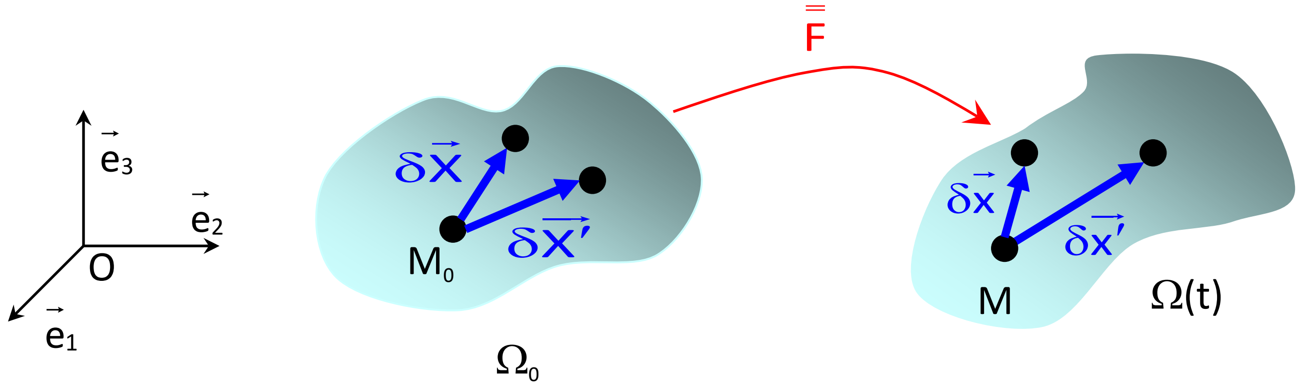




Picture from Jean Salençon course, (PhD of Le Douaron, 1977, CEMEF)

Deformation of continuous medium

Strain tensor



$$\delta\vec{x}\cdot\delta\vec{x}' - \delta\vec{X}\cdot\delta\vec{X}' = \delta\vec{x}^T\delta\vec{x}' - \delta\vec{X}^T\delta\vec{X}'$$
$$= \left(\bar{\bar{F}}\delta\vec{X}\right)^T\left(\bar{\bar{F}}\delta\vec{X}'\right)\delta\vec{X}^T\delta\vec{X}' - \delta\vec{X}^T\delta\vec{X}'$$
$$= \delta\vec{X}^T\bar{\bar{F}}^T\bar{\bar{F}}\delta\vec{X}' - \delta\vec{X}^T\delta\vec{X}'$$
$$= \delta\vec{X}^T\left(\bar{\bar{F}}^T\bar{\bar{F}} - \bar{\bar{I}}\right)\delta\vec{X}'$$

$$\delta\vec{x}\cdot\delta\vec{x}' - \delta\vec{X}\cdot\delta\vec{X}' = \delta x_i\delta x'_j\delta_{ij} - \delta X_i\delta X'_j\delta_{ij}$$
$$= F_{ik}\delta X_k F_{jn}\delta X'_n\delta_{ij} - \delta X_i\delta X'_j\delta_{ij}$$
$$= \delta X_i F_{pi}F_{qj}\delta X'_j\delta_{pq} - \delta X_i\delta X'_j\delta_{ij}$$
$$= \delta X_i\left(F_{pi}F_{qj}\delta_{pq} - \delta_{ij}\right)\delta X'_j$$
$$= \delta X_i\left(F_{pi}F_{pj} - \delta_{ij}\right)\delta X'_j$$
$$= \delta\vec{X}^T\left(\bar{\bar{F}}^T\bar{\bar{F}} - \bar{\bar{I}}\right)\delta\vec{X}'$$

We thus define the Green-Lagrange strain tensor

$$\bar{\bar{\varepsilon}} = \frac{1}{2}\left(\bar{\bar{F}}^T\bar{\bar{F}} - \bar{\bar{I}}\right)$$

Rmk: $\bar{\bar{\varepsilon}}$ is symetrical

Kinematics

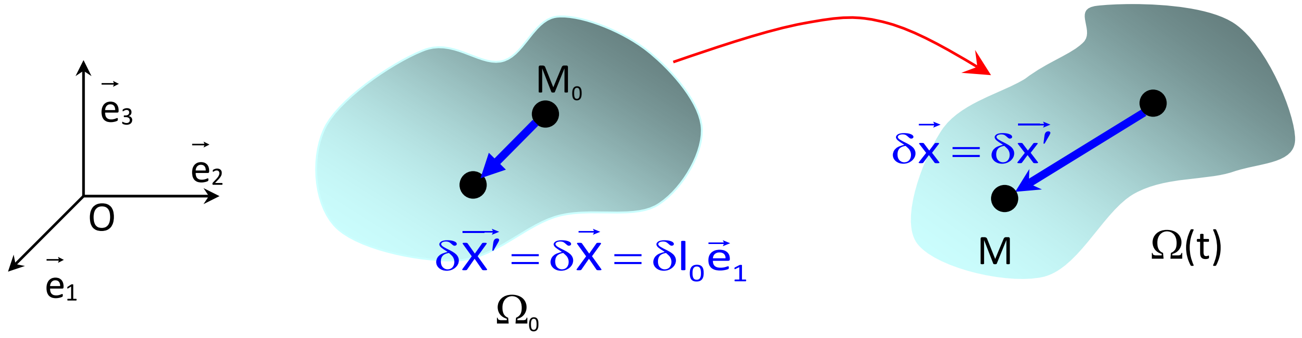
The movement and its representations

Deformation of continuous medium

Transport, material derivative

Deformation of continuous medium

More physical approach…



$$\delta\vec{x}\cdot\delta\vec{x}' - \delta\vec{X}\cdot\delta\vec{X}' = \delta l^2 - \delta l_0^2 = 2\,\delta\vec{X}^T\,\bar{\bar{\varepsilon}}\,\delta\vec{X}' = 2\langle \delta l_0 \quad 0 \quad 0 \rangle \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{Bmatrix} \delta l_0 \\ 0 \\ 0 \end{Bmatrix} = 2\delta l_0^2 \varepsilon_{11}$$

or, if strain is small

$$\frac{\delta l}{\delta l_0} = \sqrt{1+2\varepsilon_{11}} \approx 1+\varepsilon_{11} \rightarrow \varepsilon_{11} \approx \frac{\delta l - \delta l_0}{\delta l_0}$$

Variation of length in direction 1

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

Deformation of continuous medium

Assumption of small strain

Let $\vec{u}$ be the displament

$$\vec{x} = \vec{X} + \vec{u}$$

$$\overline{\overline{F}} = \overline{\overline{1}} + \nabla\vec{u}$$

$$\overline{\overline{\varepsilon}} = \frac{1}{2}\left(\nabla^T\vec{u} + \nabla\vec{u} + \nabla^T\vec{u}.\nabla\vec{u}\right)$$

Linear part of strain tensor $\left\|\nabla\vec{u}\right\| << 1$

$$\overline{\overline{\varepsilon}}_{SST} = \frac{1}{2}\left(\nabla^T\vec{u} + \nabla\vec{u}\right)$$

$$\overline{\overline{\varepsilon}}_{SST} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & .. & .. \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{\partial u_2}{\partial X_2} & .. \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial X_2} + \frac{\partial u_2}{\partial X_3}\right) & \frac{\partial u_3}{\partial X_3} \end{bmatrix}$$

Linearized Strain Tensor or Small Strain Tensor

Kinematics

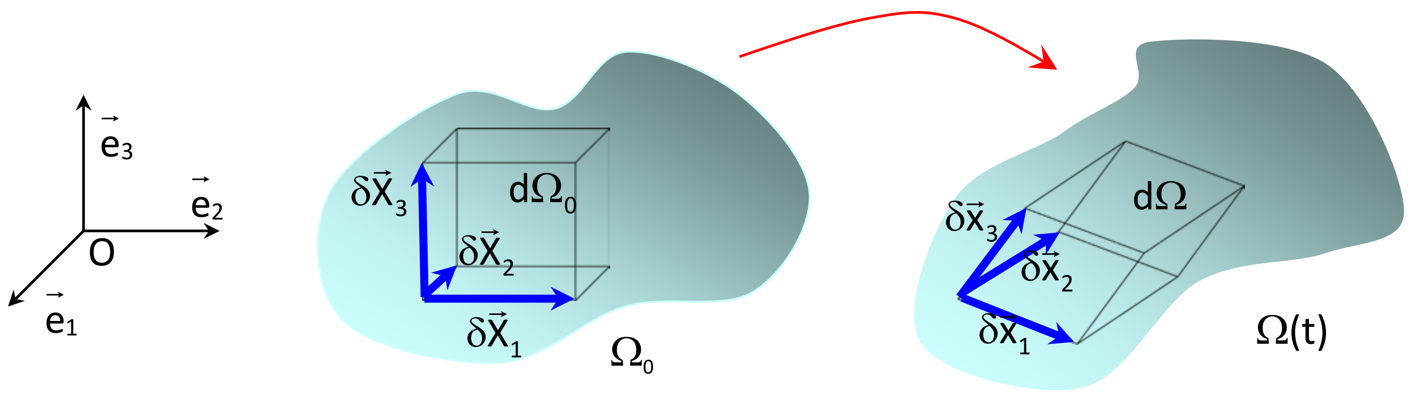
The movement and its representations

Deformation of continuous medium

Transport, material derivative

Transport, material derivative

Transport of an elementary material volume



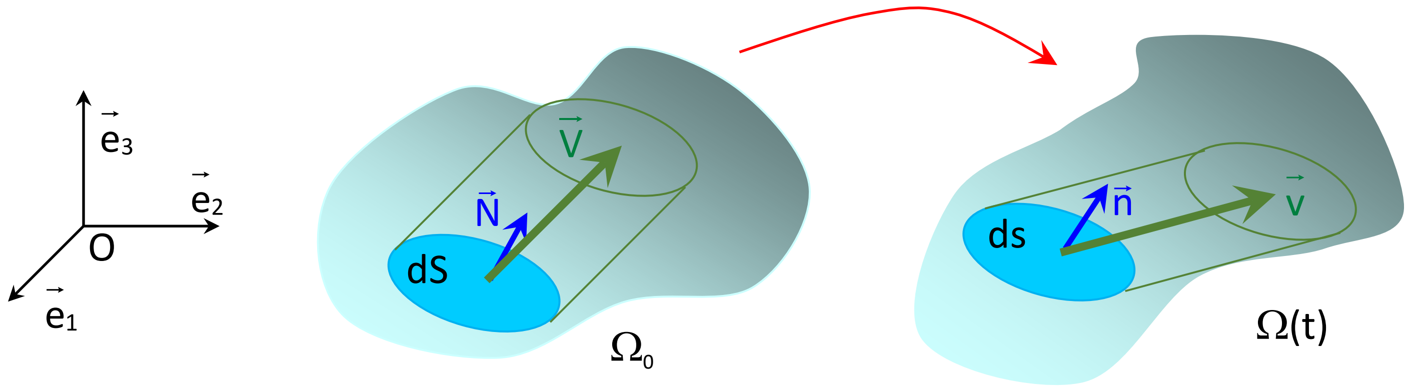
$$
\begin{aligned}
d\Omega &= \left(\delta\vec{x}_1 \wedge \delta\vec{x}_2\right)\cdot\delta\vec{x}_3 \\
&= \varepsilon_{ijk}\delta x_{1i}\delta x_{2j}\delta x_{3k} \\
&= \varepsilon_{ijk}\left(F_{ip}\delta X_{1p}\right)\left(F_{jq}\delta X_{2q}\right)\left(F_{kr}\delta X_{3r}\right) \\
&= \varepsilon_{ijk}F_{ip}F_{jq}F_{kr}\delta X_{1p}\delta X_{2q}\delta X_{3r} \\
&= \varepsilon_{pqr}\det\left(\bar{\bar{F}}\right)\delta X_{1p}\delta X_{2q}\delta X_{3r} \\
&= \det\left(\bar{\bar{F}}\right)\left(\delta\vec{X}_1 \wedge \delta\vec{X}_2\right)\cdot\delta\vec{X}_3
\end{aligned}
$$

$$d\Omega = \det\left(\bar{\bar{F}}\right)d\Omega_0 = J\,d\Omega_0$$

Rmq: For an incompressible medium $\det\left(\bar{\bar{F}}\right) = 1$

Transport, material derivative

Transport of an elementary material surface



$$ds\,\vec{n}\cdot\vec{v} = \det(\bar{\bar{F}})\,dS\,\vec{N}\cdot\vec{V}$$

$$ds\,\vec{n}\cdot\left(\bar{\bar{F}}\vec{V}\right) = \det(\bar{\bar{F}})\,dS\,\vec{N}\cdot\vec{V}$$

$$\bar{\bar{F}}^T ds\,\vec{n}\cdot\vec{V} = \det(\bar{\bar{F}})\,dS\,\vec{N}\cdot\vec{V}$$

$$ds\,\vec{n}\cdot\vec{V} = \det(\bar{\bar{F}})\,dS\,\bar{\bar{F}}^{-T}\vec{N}\cdot\vec{V}$$

$$ds\,\vec{n} = \det(\bar{\bar{F}})\,dS\,\bar{\bar{F}}^{-T}\vec{N}$$

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

Transport, material derivative

Time derivative of a volume integral

$$\frac{d}{dt}\iiint_{\Omega(t)} k(\vec{x},t)\, d\Omega = ?$$

$$\frac{d}{dt}\iiint_{\Omega(t)} k(\vec{x},t)\, d\Omega = \frac{d}{dt}\iiint_{\Omega_0} k(\vec{X},t)\, J\, d\Omega_0$$

$$= \iiint_{\Omega_0}\left(\frac{dk}{dt}J + k\frac{dJ}{dt}\right) d\Omega_0 \qquad \frac{dJ}{dt} = ?$$

$$J = \det\overline{\overline{F}} = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}F_{ip}F_{jq}F_{kr}$$

$$\frac{dJ}{dt} = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr}\frac{\partial F_{ip}}{\partial t}F_{jq}F_{kr}$$

$$\frac{\partial F_{ip}}{\partial t} = \frac{\partial^2 x_i(\vec{X},t)}{\partial t\,\partial X_p} = \frac{\partial}{\partial X_p}\left(V_i(\vec{X},t)\right) = \frac{\partial}{\partial X_p}\left(v_i(\vec{x},t)\right) = \frac{\partial v_i}{\partial x_l}\frac{\partial x_l}{\partial X_p} = \frac{\partial v_i}{\partial x_l}F_{lp}$$

$$\frac{dJ}{dt} = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr}\frac{\partial v_i}{\partial x_l}F_{lp}F_{jq}F_{kr} = \frac{1}{2}\varepsilon_{ijk}\frac{\partial v_i}{\partial x_l}\varepsilon_{ljk}\det\left(\overline{\overline{F}}\right) = \frac{1}{2}2\delta_{il}\frac{\partial v_i}{\partial x_l}J = \frac{\partial v_i}{\partial x_i}J = J\,\mathrm{div}\left(\vec{v}\right)$$

$$\frac{d}{dt}\iiint_{\Omega(t)} k(\vec{x},t)\, d\Omega = \iiint_{\Omega_0}\left(\frac{dk}{dt}J + k\,J\,\mathrm{div}\vec{v}\right) d\Omega_0$$

$$= \iiint_{\Omega_0}\left(\frac{dk}{dt} + k\,\mathrm{div}\vec{v}\right) J\, d\Omega_0$$

$$\frac{d}{dt}\iiint_{\Omega(t)} k(\vec{x},t)\, d\Omega = \iiint_{\Omega(t)}\left(\frac{dk}{dt} + k\,\mathrm{div}\vec{v}\right) d\Omega$$

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

Transport, material derivative

Time derivative of a volume integral : important application!

$$M = \iiint_{\Omega(t)} \rho(\vec{x}, t)\, d\Omega$$

$$\frac{dM}{dt} = 0$$ → Mass conservation

$$\frac{d\rho}{dt} + \rho\, \mathrm{div}\vec{v} = 0$$

or

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0$$

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

Transport, material derivative

Time derivative of volume integral : further applications

$$\frac{d}{dt}\iiint_{\Omega(t)} k(\vec{x},t)\, d\Omega = \iiint_{\Omega(t)}\left(\frac{dk}{dt} + k\, \mathrm{div}\vec{v}\right) d\Omega$$

$$= \iiint_{\Omega(t)}\left(\frac{\partial k}{\partial t} + \vec{v}\cdot\nabla k + k\, \mathrm{div}\vec{v}\right) d\Omega$$

$$= \iiint_{\Omega(t)}\left(\frac{\partial k}{\partial t} + \mathrm{div}\left(k\vec{v}\right)\right) d\Omega$$

$$\frac{d}{dt}\iiint_{\Omega(t)} \vec{A}(\vec{x},t)\, d\Omega = \iiint_{\Omega(t)}\left(\frac{\partial \vec{A}}{\partial t} + \mathrm{div}\left(\vec{A}\otimes\vec{v}\right)\right) d\Omega$$

$$\frac{d}{dt}\iiint_{\Omega(t)} k(\vec{x},t)\, \rho d\Omega = \iiint_{\Omega(t)}\left(\frac{d\rho k}{dt} + \rho k\, \mathrm{div}\vec{v}\right) d\Omega$$

$$= \iiint_{\Omega(t)}\left(\rho\frac{dk}{dt} + k\frac{d\rho}{dt} + \rho k\, \mathrm{div}\vec{v}\right) d\Omega$$

$$= \iiint_{\Omega(t)} \frac{dk}{dt}\rho d\Omega$$

$$\frac{d}{dt}\iiint_{\Omega(t)} {*}\, \rho d\Omega = \iiint_{\Omega(t)} \frac{d*}{dt}\rho d\Omega$$

Kinematics

The movement and its representations

Deformation of continuous medium

Transport, material derivative

Transport, material derivative

Time derivative of a surface integral

$$\frac{d}{dt}\iint_{\Sigma(t)}\vec{k}(\vec{x},t)\cdot\vec{n}\,d\Sigma = ?$$

$$\frac{d}{dt}\iint_{\Sigma(t)}\vec{k}(\vec{x},t)\cdot\vec{n}\,d\Sigma = \frac{d}{dt}\iint_{\Sigma_0}\vec{k}\left(\vec{X},t\right)\cdot J\,\overline{\overline{F}}^{-T}\vec{N}d\Sigma_0$$

$$= \iint_{\Sigma_0}\left(\frac{d\vec{k}}{dt}\cdot J\,\overline{\overline{F}}^{-T}\vec{N} + \vec{k}\cdot\frac{dJ}{dt}\overline{\overline{F}}^{-T}\vec{N} + \vec{k}\cdot J\frac{d\overline{\overline{F}}^{-T}}{dt}\vec{N}\right)d\Sigma_0 \qquad \frac{d\overline{\overline{F}}^{-T}}{dt} = ?$$

$$\overline{\overline{F}}^{-1}\overline{\overline{F}} = \overline{\overline{I}} \Rightarrow \frac{d\overline{\overline{F}}^{-1}}{dt}\overline{\overline{F}} + \overline{\overline{F}}^{-1}\frac{d\overline{\overline{F}}}{dt} = \overline{\overline{0}} \Rightarrow \frac{d\overline{\overline{F}}^{-1}}{dt} = -\overline{\overline{F}}^{-1}\frac{d\overline{\overline{F}}}{dt}\overline{\overline{F}}^{-1}$$

$$\frac{d\overline{\overline{F}}}{dt}\overline{\overline{F}}^{-1} = \frac{\partial v_i}{\partial X_k}\frac{\partial X_k}{\partial x_j}\vec{e}_i\otimes\vec{e}_j = \frac{\partial v_i}{\partial x_j}\vec{e}_i\otimes\vec{e}_j = \nabla\vec{v} \qquad \frac{d\overline{\overline{F}}^{-T}}{dt} = -\left(\overline{\overline{F}}^{-1}\nabla\vec{v}\right)^T = -\nabla^T\vec{v}\,\overline{\overline{F}}^{-T}$$

$$\frac{d}{dt}\iint_{\Sigma(t)}\vec{k}(\vec{x},t)\cdot\vec{n}\,d\Sigma = \iint_{\Sigma_0}\left(\frac{d\vec{k}}{dt}\cdot J\,\overline{\overline{F}}^{-T}\vec{N} + \vec{k}\cdot J\,\mathrm{div}\vec{v}\,\overline{\overline{F}}^{-T}\vec{N} - \vec{k}\cdot J\,\nabla^T\vec{v}\,\overline{\overline{F}}^{-T}\vec{N}\right)d\Sigma_0$$

$$= \iint_{\Sigma_0}\left(\frac{d\vec{k}}{dt} + \vec{k}\,\mathrm{div}\vec{v} - \vec{k}\nabla^T\vec{v}\right)\cdot J\,\overline{\overline{F}}^{-T}\vec{N}d\Sigma_0$$

$$\frac{d}{dt}\iint_{\Sigma(t)}\vec{k}(\vec{x},t)\cdot\vec{n}\,d\Sigma = \iint_{\Sigma(t)}\left(\frac{d\vec{k}}{dt} + \vec{k}\,\mathrm{div}\vec{v} - \vec{k}\nabla^T\vec{v}\right)\cdot\vec{n}\,d\Sigma$$

Stress in continuous medium

Notion of stress

Balance of momentum

Some properties of the stress tensor

Lecture on Continuum Mechanics
SeaTech 1st year

Part 3

Stress in continuous medium

F. Golay

Stress in continuous medium

Notion of stress

Balance of momentum

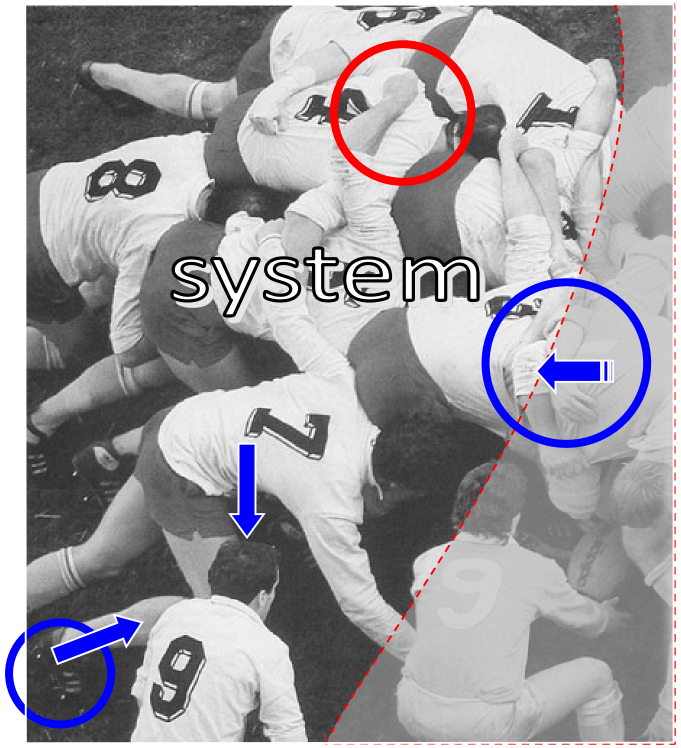
Some properties of the stress tensor

Notion of Stress

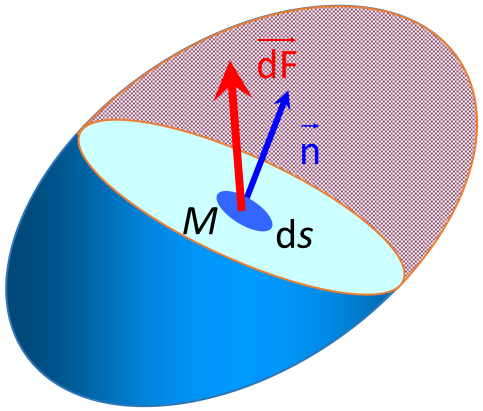
Notion of stress

External forces

Internal forces



Picture from *Le Rugby*
P. VILLEPREUX
Lecture of J.Salençon
Polytechnique



"Cohésion" strength $\overrightarrow{dF}(\vec{x},t,\vec{n},ds)$ [N]

$$\overrightarrow{dF}(\vec{x},t,\vec{n},ds)=\vec{T}(\vec{x},t,\vec{n})ds$$

$\vec{T}$: Stress vector [N/m²]

$$\vec{T}(\vec{x},t,\vec{n})=\overline{\overline{\sigma}}(\vec{x},t)\vec{n}$$

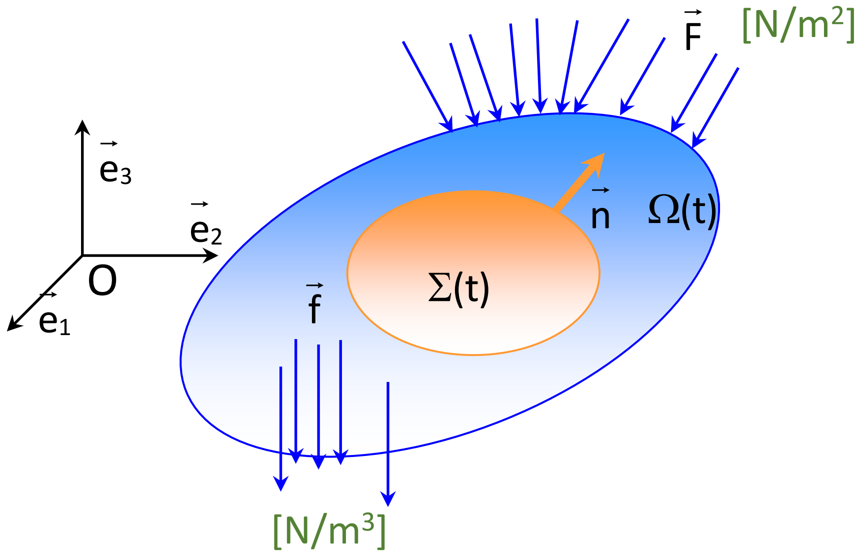
$\overline{\overline{\sigma}}(\vec{x},t)$ Cauchy stress tensor

[N/m²]

Stress in continuous medium

Balance of momentum

Fundamental principle of dynamics



Fundamental principle of dynamics

Dynamical torsor
=
External forces torsor

$$\begin{cases} \dfrac{d}{dt}\iiint_{\Sigma} \rho \vec{v}\, dV = \iiint_{\Sigma} \vec{f}\, dV + \iint_{\partial\Sigma} \vec{T}\, ds \\ \dfrac{d}{dt}\iiint_{\Sigma} \rho \overrightarrow{OM} \wedge \vec{v}\, dV = \iiint_{\Sigma} \overrightarrow{OM} \wedge \vec{f}\, dV + \iint_{\partial\Sigma} \overrightarrow{OM} \wedge \vec{T}\, ds \end{cases}$$

Stress in continuous medium

Notion of stress

Balance of momentum

Some properties of the stress tensor

Balance of momentum

Local formulation of balance of linear momentum

$$\frac{d}{dt}\iiint_\Sigma \rho\vec{v}\, dV = \iiint_\Sigma \vec{f}\, dV + \iint_{\partial\Sigma} \vec{T}\, ds$$

$$\iiint_\Sigma \rho\frac{d\vec{v}}{dt}\, dV = \iiint_\Sigma \vec{f}\, dV + \iint_{\partial\Sigma} \overline{\overline{\sigma}}\vec{n}\, ds$$

Mass conservation

Definition of stress vector

$$\iiint_\Sigma \rho\frac{d\vec{v}}{dt}\, dV = \iiint_\Sigma \vec{f}\, dV + \iiint_\Sigma \mathrm{div}\left(\overline{\overline{\sigma}}\right) dV$$

Green theorem

$$\forall\Sigma \quad \iiint_\Sigma \left(\mathrm{div}\overline{\overline{\sigma}} + \vec{f} - \rho\vec{\gamma}\right) dV = \vec{0}$$

Local formulation of balance of linear momentum

$$\begin{cases} \mathrm{div}\overline{\overline{\sigma}} + \vec{f} = \rho\vec{\gamma} & \text{in } \Omega \quad [N/m^3] \\ \overline{\overline{\sigma}}\vec{n} = \vec{F} & \text{on } \partial\Omega \quad [N/m^2] \end{cases}$$

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Stress in continuous medium

Notion of stress

Balance of momentum

Some properties of the stress tensor

Balance of momentum

Local formulation of balance of angular momentum

$$\frac{d}{dt}\iiint_\Sigma \rho \overrightarrow{OM}\wedge\vec{v}\,dV = \iiint_\Sigma \overrightarrow{OM}\wedge\vec{f}\,dV + \iint_{\partial\Sigma}\overrightarrow{OM}\wedge\vec{T}\,ds$$

No external density of moment!

$$\iiint_\Sigma \rho\frac{d}{dt}\left(\overrightarrow{OM}\wedge\vec{v}\right)dV = \iiint_\Sigma \overrightarrow{OM}\wedge\vec{f}\,dV + \iint_{\partial\Sigma}\overrightarrow{OM}\wedge\vec{T}\,ds$$

$$\iiint_\Sigma \rho\vec{v}\wedge\vec{v}\,dV + \iiint_\Sigma \rho\overrightarrow{OM}\wedge\vec{\gamma}\,dV = \iiint_\Sigma \overrightarrow{OM}\wedge\vec{f}\,dV + \iint_{\partial\Sigma}\overrightarrow{OM}\wedge\overline{\overline{\sigma}}\vec{n}\,ds$$

$$\iiint_\Sigma \varepsilon_{ijk}x_j(\rho\gamma_k - f_k)\,\vec{e}_i\,dx = \iint_{\partial\Sigma}\varepsilon_{ijk}x_j\sigma_{kl}n_l\,\vec{e}_i\,dx$$

Index notations

$$\iiint_\Sigma \left[\varepsilon_{ijk}x_j(\rho\gamma_k - f_k) - \frac{\partial}{\partial x_p}(\varepsilon_{ijk}x_j\sigma_{kp})\right]\vec{e}_i\,dx = 0$$

Green theorem

$$\iiint_\Sigma \left[\varepsilon_{ijk}x_j(\rho\gamma_k - f_k - \sigma_{kp,p}) - \varepsilon_{ijk}\sigma_{kp}\delta_{pj}\right]\vec{e}_i\,dx = 0$$

Balance of linear momentum

$$\forall\Sigma \quad \iiint_\Sigma \varepsilon_{ijk}\sigma_{kj}\,\vec{e}_i\,dx = 0$$

$$\varepsilon_{ijk}\sigma_{kj}\,\vec{e}_i = \vec{0}$$

$$\varepsilon_{123}\sigma_{23} + \varepsilon_{132}\sigma_{32} = 0 \qquad +\sigma_{23} - \sigma_{32} = 0$$

$$\varepsilon_{213}\sigma_{13} + \varepsilon_{231}\sigma_{31} = 0 \qquad -\sigma_{13} + \sigma_{31} = 0$$

$$\varepsilon_{312}\sigma_{12} + \varepsilon_{321}\sigma_{21} = 0 \qquad +\sigma_{12} - \sigma_{21} = 0$$

$\overline{\overline{\sigma}}$ is symetric!

Stress in continuous medium

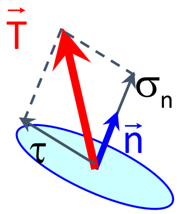
Notion of stress

Balance of momentum

Some properties of the stress tensor

Some properties of the stress tensor

Normal and tangential stresses



σ_n Normal stress
τ Tangential stress

Principal stresses and principal directions

$\bar{\bar{\sigma}}$ is symetric $\Rightarrow$ $\bar{\bar{\sigma}}$ is diagonalizable

Eigenvalues : principal stresses
Eigenvectors : principal directions

$$\bar{\bar{\sigma}} = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

Invariants

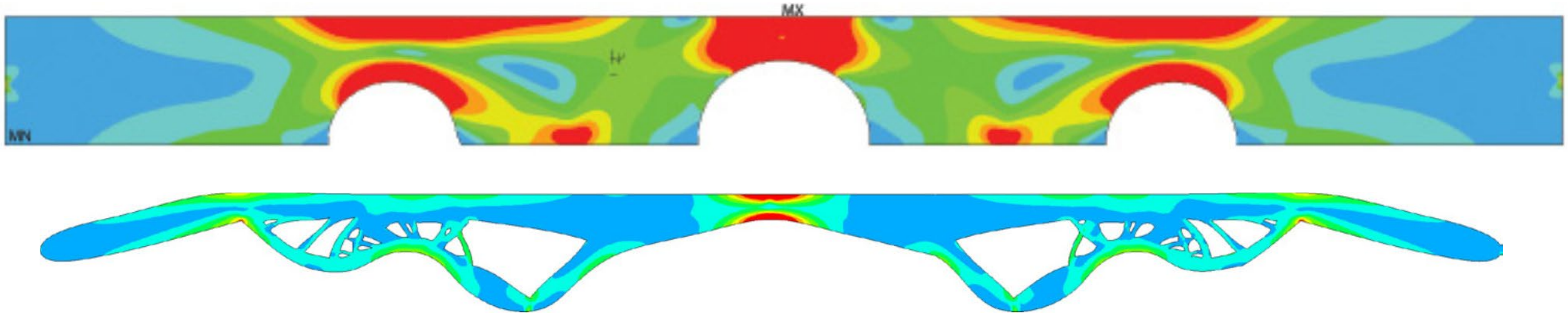
$$\Sigma_I = Tr(\bar{\bar{\sigma}}) = \sigma_I + \sigma_{II} + \sigma_{III}$$

$$\Sigma_{II} = \frac{1}{2}\left[Tr(\bar{\bar{\sigma}})^2 - Tr(\bar{\bar{\sigma}}^2)\right] = \sigma_I\sigma_{II} + \sigma_{II}\sigma_{III} + \sigma_{III}\sigma_I$$

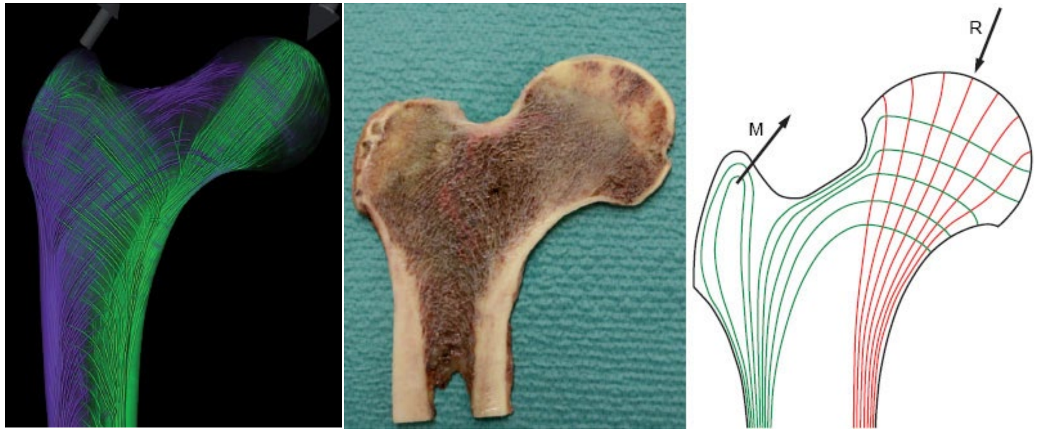
$$\Sigma_{III} = Det(\bar{\bar{\sigma}}) = \sigma_I\sigma_{II}\sigma_{III}$$

Some properties of the stress tensor

Illustrations



“Topology Optimisation for Steel Structural Design with Additive Manufacturing”, Ren S., Galjaard S., Conference: Design Modelling Symposium 2015, Copenhagen, doi=10.1007/978-3-319-24208-8_3



Christian Dick, Joachim Georgii, Rainer Burgkart, and Rüdiger Westermann. Stress tensor field visualization for implant planning in orthopedics. IEEE Transactions on Visualization and Computer Graphics (Proceedings of IEEE Visualization 2009), 15(6):1399–1406, 2009.

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Stress in continuous medium

Notion of stress

Balance of momentum

Some properties of the stress tensor

Some properties of the stress tensor

Mohr's circles

$\overline{\overline{\sigma}}$ given

$\vec{n}$ varying

$\vec{T}$??

$$\vec{n} = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}, \quad \overline{\overline{\sigma}} = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix} \quad \text{et} \quad \vec{T} = \begin{Bmatrix} n_1\,\sigma_I \\ n_2\,\sigma_{II} \\ n_3\,\sigma_{III} \end{Bmatrix}$$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

$$\sigma_n = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2$$

$$\tau^2 + \sigma_n^2 = \sigma_I^2 n_1^2 + \sigma_{II}^2 n_2^2 + \sigma_{III}^2 n_3^2$$

$$n_1^2 = \frac{\tau^2 + (\sigma_n - \sigma_{II})(\sigma_n - \sigma_{III})}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})}$$

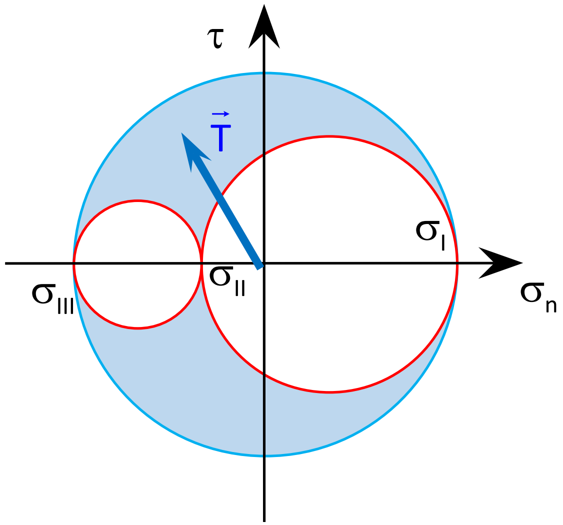
$$n_2^2 = \frac{\tau^2 + (\sigma_n - \sigma_I)(\sigma_n - \sigma_{III})}{(\sigma_{II} - \sigma_I)(\sigma_{II} - \sigma_{III})}$$

$$n_3^2 = \frac{\tau^2 + (\sigma_n - \sigma_I)(\sigma_n - \sigma_{II})}{(\sigma_{III} - \sigma_I)(\sigma_{III} - \sigma_{II})}$$

$$\tau^2 + \left(\sigma_n - \frac{\sigma_{II} + \sigma_{III}}{2}\right)^2 \geq \left(\frac{\sigma_{II} - \sigma_{III}}{2}\right)^2$$

$$\tau^2 + \left(\sigma_n - \frac{\sigma_I + \sigma_{III}}{2}\right)^2 \leq \left(\frac{\sigma_I - \sigma_{III}}{2}\right)^2$$

$$\tau^2 + \left(\sigma_n - \frac{\sigma_I + \sigma_{II}}{2}\right)^2 \geq \left(\frac{\sigma_I - \sigma_{II}}{2}\right)^2$$



$$\tau_{max} = \left| \frac{\sigma_I - \sigma_{III}}{2} \right|$$

Stress in continuous medium

Notion of stress

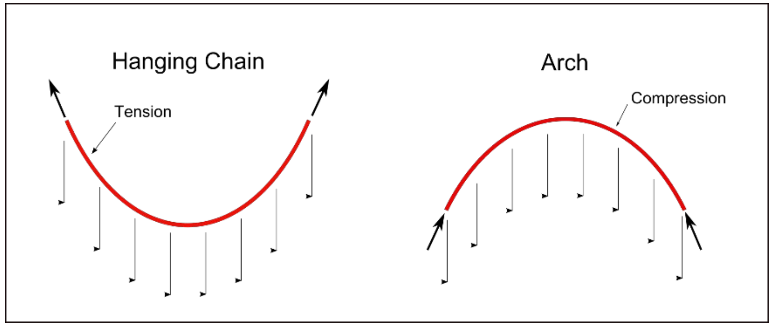
Balance of momentum

Some properties of the stress tensor

Illustration : topological optimisation, arc, …

Funicular curve - catenary

Robert Hooke (1635-1703): «*Ut pendet continuum flexile, sic stabit contiguum rigidum inversum* »
« Just as a flexible thread hangs, so, in reverse, we find the contiguous parts of an arch. »



Source wikipedia :https://fr.wikipedia.org/wiki/Cha%C3%AEnette

The arch of Ctésiphon from the palace of Taq-i Kisra, near Baghdad in Iraq (531 after J.C.).(source https://artkarel.com/tag/chainette/)

The attic of the Ramasséum, near Thebes in Egypt (XIII^{ème} century before J.-C.). (source https://artkarel.com/tag/chainette/)

Stress in continuous medium

Notion of stress

Balance of momentum

Some properties of the stress tensor

Stress in continuous medium

Notion of stress

Balance of momentum

Some properties of the stress tensor

Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Lecture on Continuum Mechanics
SeaTech 1st year

Part 4

Elasticity, thermoelasticity and introduction to fluid mechanics

F. Golay

Elasticity …

Introduction

Constitutive equation

$$\Re\left(\overline{\overline{\sigma}},\overline{\overline{\varepsilon}},\frac{d\overline{\overline{\sigma}}}{dt},\frac{d\overline{\overline{\varepsilon}}}{dt},T,\alpha,\ldots\right)=0$$

Recommended reading:

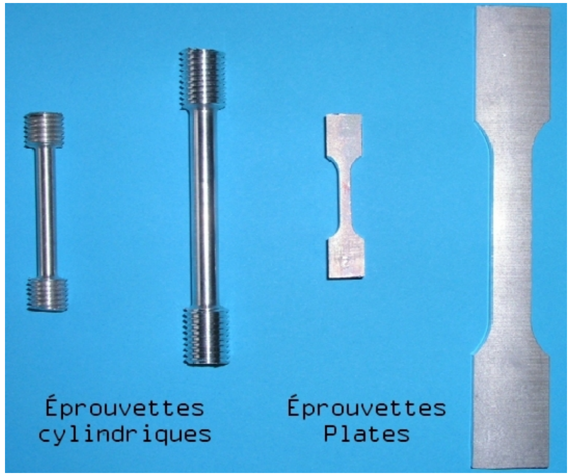
P. Germain, *Mécanique*, ed. ellipse, école polytechnique, tomes I et II

J. Lemaitre, J.L. Chaboche, *Mécanique des matériaux solides*, ed. Dunos 1996

Jean Coirier : « *Mécanique des Milieux Continus* », ed. Dunod, Paris, 2001.

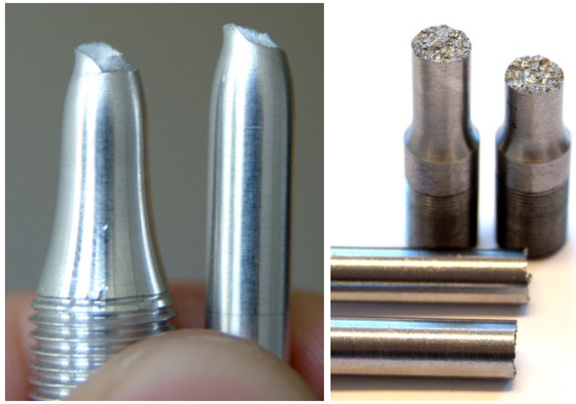
Constitutive equation

Tensile test



Example of tensile test specimens
(source Wikipédia)

Tensile test machine SIM/UTLN



Example of rupture of tensile test specimens
(source Wikipédia)

Elasticity …

Constitutive law

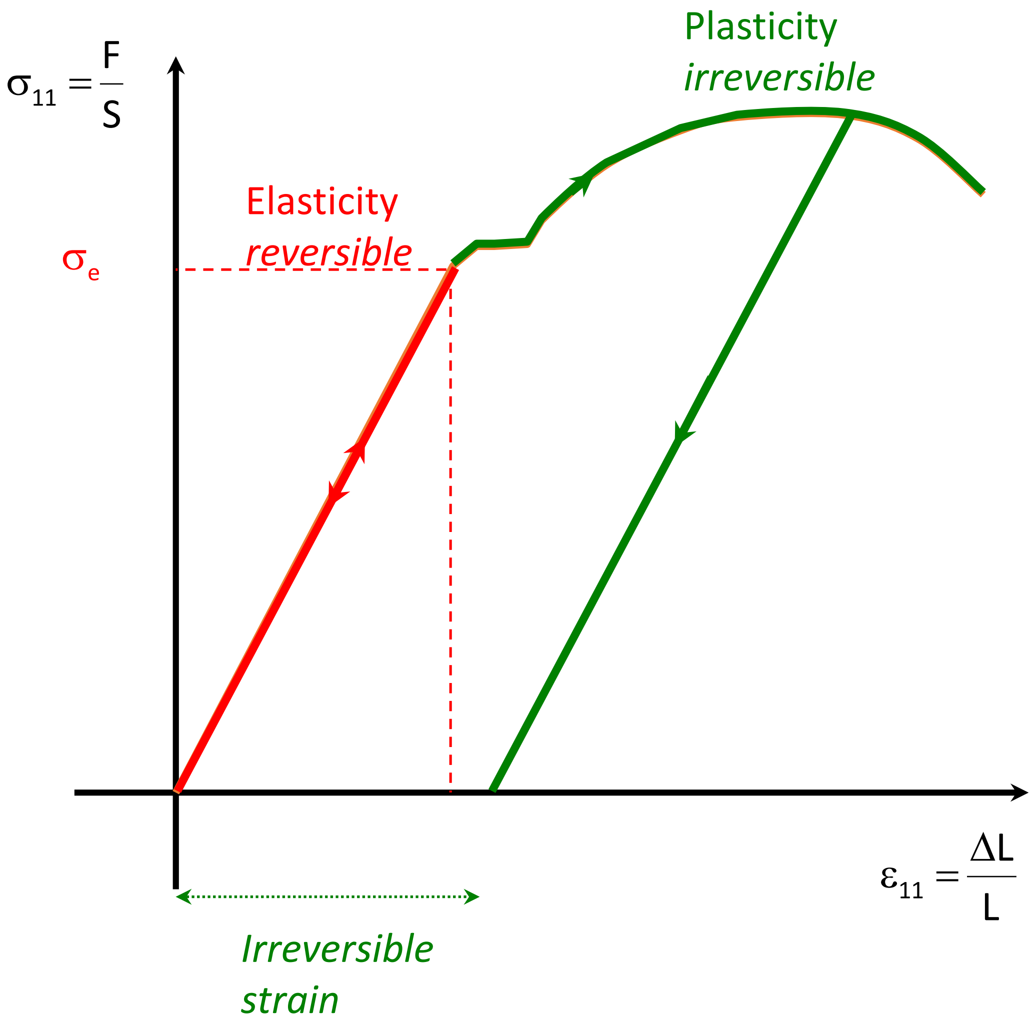
Elasticity problem

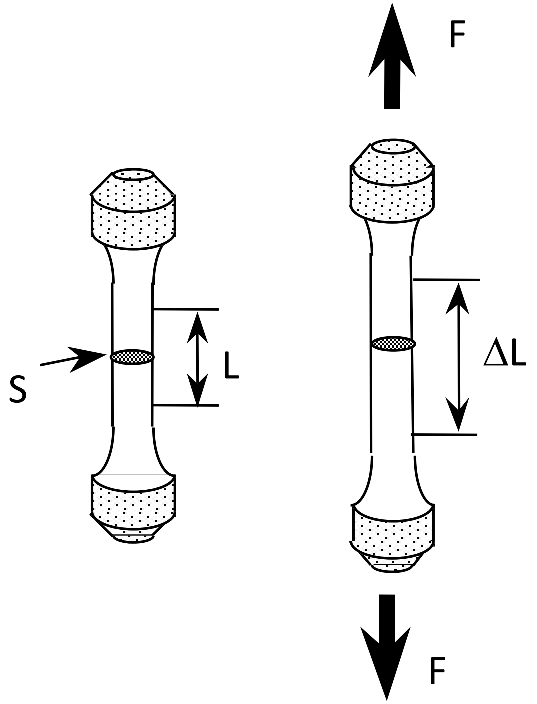
Thermoelasticity

Introduction to fluid mechanics

Constitutive equation

Tensile curve





Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Constitutive equation

General elastic constitutive equation

$$\overline{\overline{\sigma}}(\vec{x},t) = \overline{\overline{\overline{\overline{C}}}}(\vec{x}) : \overline{\overline{\varepsilon}}(\vec{x},t)$$

$$\sigma_{ij}\,\vec{e}_i \otimes \vec{e}_j = C_{ijpq}\,\varepsilon_{pq}\,\vec{e}_i \otimes \vec{e}_j \qquad C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} \Rightarrow$$ 21 coefficients

Isotropic elastic constitutive equation

$$\overline{\overline{\sigma}} = \lambda \mathrm{Tr}\left(\overline{\overline{\varepsilon}}\right)\overline{\overline{I}} + 2\mu\overline{\overline{\varepsilon}}$$

$$\sigma_{ij}\,\vec{e}_i \otimes \vec{e}_j = \left(\lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}\right)\vec{e}_i \otimes \vec{e}_j$$

σ $[N/m^2]$

ε $[-]$

λ,μ $[N/m^2]$ Lamé coefficients

$$\overline{\overline{\varepsilon}} = -\frac{\nu}{E}\mathrm{Tr}\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}} + \frac{1+\nu}{E}\overline{\overline{\sigma}}$$

$$\varepsilon_{ij}\,\vec{e}_i \otimes \vec{e}_j = \left(-\frac{\nu}{E}\sigma_{kk}\,\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}\right)\vec{e}_i \otimes \vec{e}_j$$

σ $[N/m^2]$

ε $[-]$

ν $[-]$ Poisson’s ratio

E $[N/m^2]$ Young’s modulus

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \text{et} \quad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

$$\mu = \frac{E}{2\,(1+\nu)} \quad \text{et} \quad \lambda = \frac{\nu E}{(1+\nu)\,(1-2\nu)}$$

Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

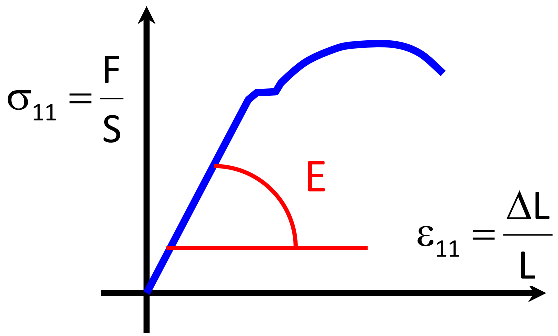
Introduction to fluid mechanics

Constitutive equation

Application to the tensile test

$$\overline{\overline{\varepsilon}} = -\frac{\nu}{E}\mathrm{Tr}\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}} + \frac{1+\nu}{E}\overline{\overline{\sigma}}$$

$$\overline{\overline{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} F/S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \overline{\overline{\varepsilon}} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\frac{\nu}{E}\sigma & 0 \\ 0 & 0 & -\frac{\nu}{E}\sigma \end{bmatrix}$$



Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Constitutive equation

Digression: Composite material, generalities

Matrix + Reinforcement = composite

- Metal
- Ceramic
- Organic
  - Thermoplastic
  - Thermosetting
  - Bio-polymeric
- …

- Glass
- Carbon
- Kevlar
- Metal
- Ceramic
- Bio
- …

Example : Cob=mud+wood!

http://www.branche-rouge.org/les-articles/tous-les-articles/artisanats/architecture-et-construction/la-construction-dun-maison-an-mil-avec-des-outils-depoque

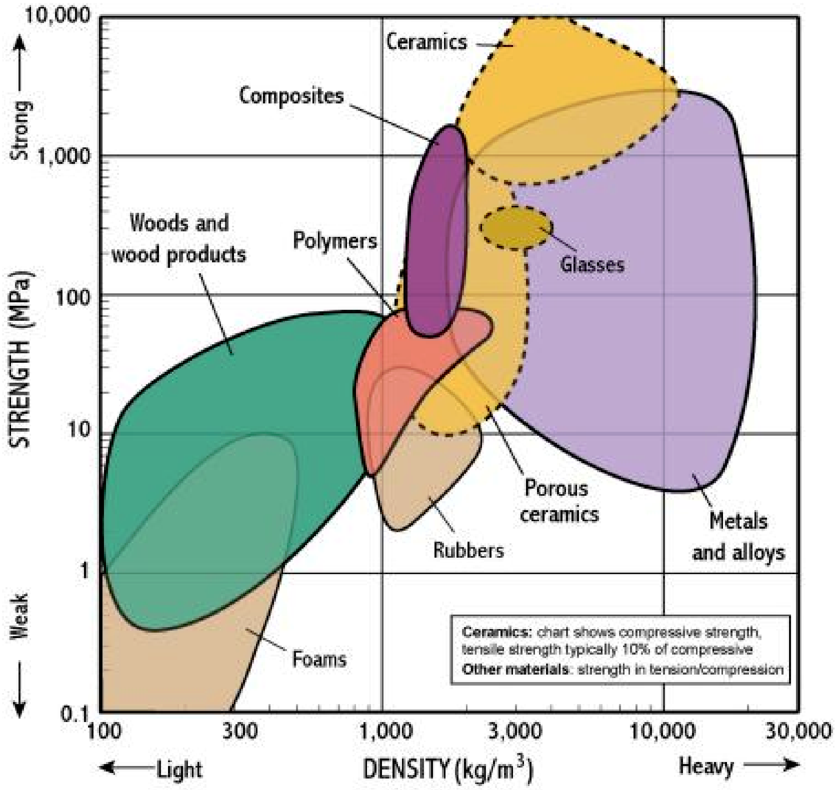
Recommended reading :

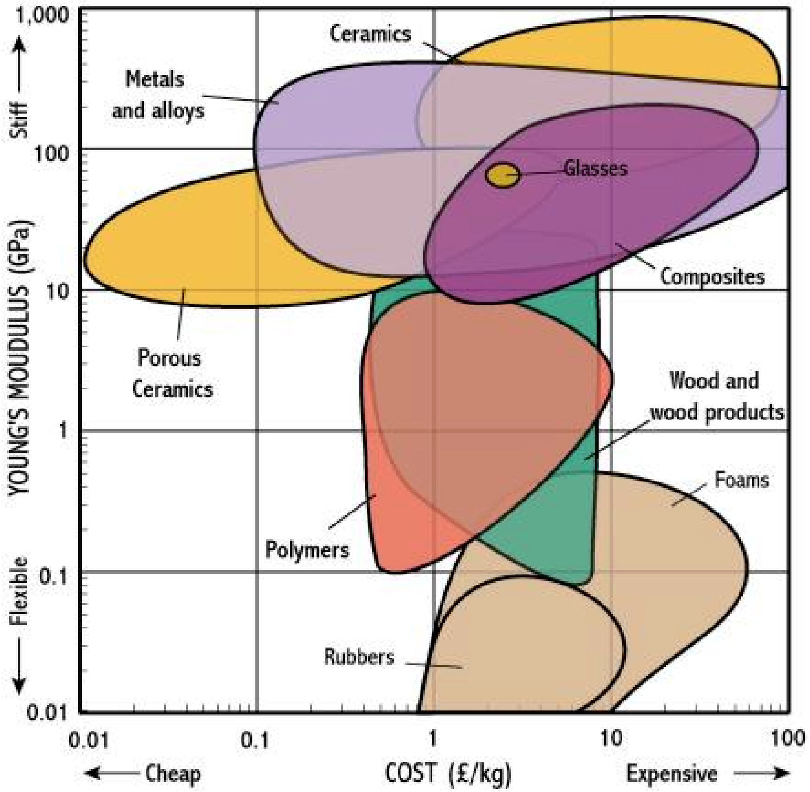
« *Matériaux composites, Comportement mécanique et analyse des structures* », BERTHELOT Jean-Marie, ed. Lavoisier.

« *Matériaux composites* », GAY Daniel, ed. Lavoisier,

Constitutive equation

Digression: Composite material, why ?





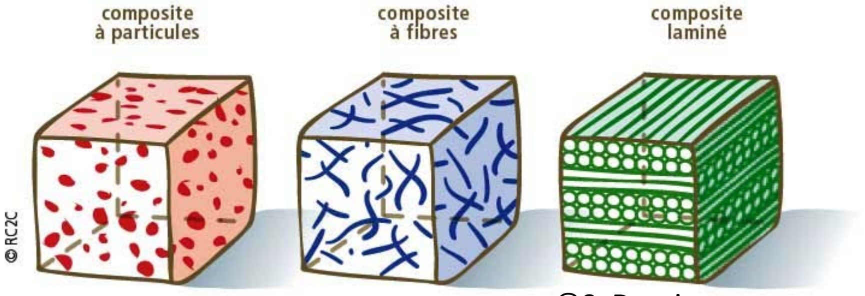
Lecture of Sylvain Drapier Mines de Saint-Etienne

http://www.emse.fr/~drapier/index_fichiers/CoursPDF/RMP-Composites/RMP-composites-SDrapier.pdf

MINES Saint-Étienne

Constitutive equation

Digression: Composite material, some kind of reinforcement



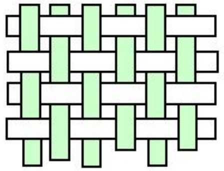
©S. Drapier

Coiled long fiber
©CEA

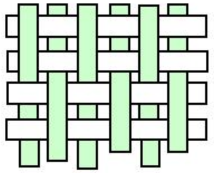
Mat : short fiber

Fabric

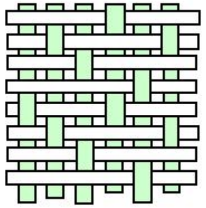
3D weaving
©Eric Drouin / Snecma / Safran



taffetas



Serge



Satin

Elasticity …

Constitutive law

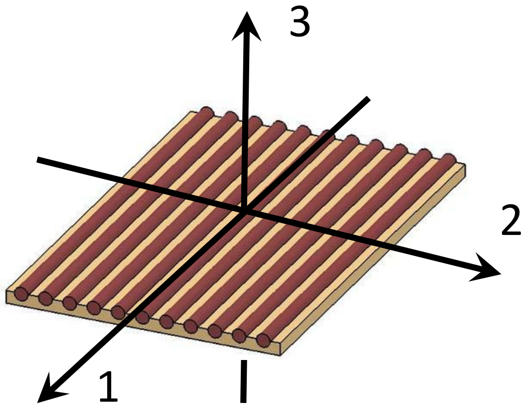
Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Constitutive equation

Orthotropic elastic constitutive equation



$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}
=
\begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\
-\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\
-\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{23}}
\end{bmatrix}
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix}
$$

Compliance matrix

Constitutive equation

Criteria of elastic limit

Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Tresca criterion

$$\frac{1}{2}\mathrm{Sup}\left\{\left|\sigma_I-\sigma_{II}\right|,\left|\sigma_I-\sigma_{III}\right|,\left|\sigma_{II}-\sigma_{III}\right|\right\} \leq \sigma_0$$

Von-Mises criterion

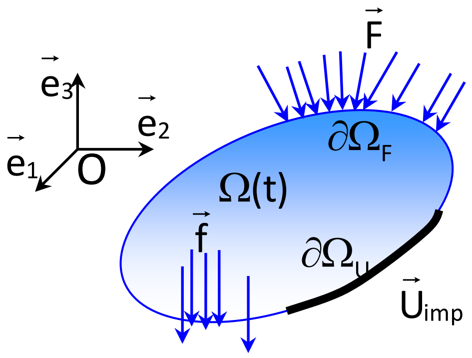
$$\sqrt{\frac{1}{2}\left((\sigma_I-\sigma_{II})^2+(\sigma_I-\sigma_{III})^2+(\sigma_{II}-\sigma_{III})^2\right)} \leq \sigma_0$$

Hill criterion

$$F(\sigma_{11}-\sigma_{22})^2+H(\sigma_{11}-\sigma_{33})^2+G(\sigma_{22}-\sigma_{33})^2+2L\sigma_{23}^2+2M\sigma_{13}^2+2N\sigma_{12}^2=1$$

Elasticity problem

General case



Small strains assumption

$$\bar{\bar{\varepsilon}} = \frac{1}{2}\left(\nabla\vec{u} + \nabla^T\vec{u}\right) \qquad [-]$$

$$\vec{u} = \vec{U}_{imp} \quad \text{on } \partial\Omega_U \qquad [m]$$

Compatibility equation

$$\mathrm{div}\,\bar{\bar{\sigma}} + \vec{f} = \rho\vec{\gamma} \quad \text{in } \Omega \qquad \left[N/m^3\right]$$

$$\bar{\bar{\sigma}}\vec{n} = \begin{cases} \vec{F} & \text{on } \partial\Omega_F \\ \vec{R} & \text{on } \partial\Omega_U \end{cases} \qquad \left[N/m^2\right]$$

$$\bar{\bar{\sigma}} = \lambda\,\mathrm{tr}\left(\bar{\bar{\varepsilon}}\right)\bar{\bar{1}} + 2\mu\bar{\bar{\varepsilon}} \qquad \left[N/m^2\right]$$

Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Elasticity problem

Displacement formulation

$$\mathrm{div}\,\overline{\overline{\sigma}} + \vec{f} = \vec{0}$$

$$\mathrm{div}(\lambda \mathrm{Tr}(\overline{\overline{\varepsilon}})\overline{\overline{I}} + 2\mu\overline{\overline{\varepsilon}}) + \vec{f} = \vec{0}$$

$$\lambda\nabla(\mathrm{Tr}(\overline{\overline{\varepsilon}})) + 2\mu\,\mathrm{div}(\overline{\overline{\varepsilon}}) + \vec{f} = \vec{0}$$

$$\lambda\nabla(\mathrm{div}\vec{u}) + \mu\,\mathrm{div}(\nabla\vec{u}) + \mu\,\mathrm{div}(\nabla^T\vec{u}) + \vec{f} = \vec{0}$$

$$\mathrm{div}\,\overline{\overline{\sigma}} = \sigma_{ij,j}\vec{e}_i$$
$$= \left(\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\right)_{,j}\vec{e}_i$$
$$= \left(\lambda u_{k,k}\delta_{ij} + \mu u_{i,j} + \mu u_{j,i}\right)_{,j}\vec{e}_i$$
$$= \lambda u_{k,ki}\vec{e}_i + \mu u_{i,jj}\vec{e}_i + \mu u_{j,ij}\vec{e}_i$$
$$= (\lambda+\mu)\left(u_{k,k}\right)_{,i}\vec{e}_i + \mu\left(u_i\vec{e}_i\right)_{,jj}$$

Navier equation

$$(\lambda+\mu)\nabla(\mathrm{div}\vec{u}) + \mu\,\mathrm{div}(\nabla\vec{u}) + \vec{f} = \vec{0}$$

Stress formulation

Michell Equation (or Beltrami equation when volumic forces are absent)

$$\mathrm{div}(\nabla\overline{\overline{\sigma}}) + \frac{1}{1+\nu}\nabla(\nabla\,\mathrm{Tr}(\overline{\overline{\sigma}})) + \frac{\nu}{1-\nu}\mathrm{div}\vec{f}\,\overline{\overline{I}} + \nabla\vec{f} + \nabla^T\vec{f} = \overline{\overline{0}}$$

Elasticity …

Elasticity problem

Two-dimensional linear elasticity with plane stress

Thin structures along direction 3

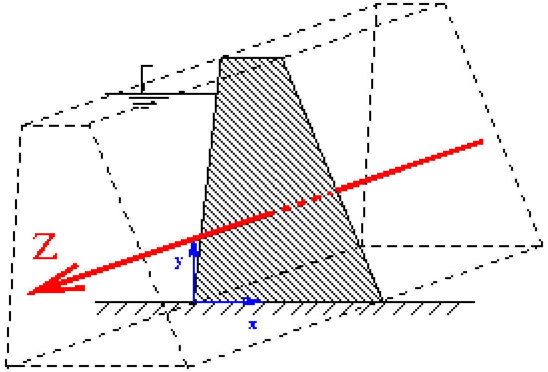
$$\overline{\overline{\sigma}} = \begin{bmatrix} \sigma_{11}(x_1,x_2) & \sigma_{12}(x_1,x_2) & 0 \\ \sigma_{21}(x_1,x_2) & \sigma_{22}(x_1,x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_{11}(x_1,x_2) & \varepsilon_{11}(x_1,x_2) & 0 \\ \varepsilon_{21}(x_1,x_2) & \varepsilon_{22}(x_1,x_2) & 0 \\ 0 & 0 & \varepsilon_{33}(x_1,x_2) \end{bmatrix}$$

Two-dimensional linear elasticity with plane strain

Slender structures alongdirection 3

$$\overline{\overline{\sigma}} = \begin{bmatrix} \sigma_{11}(x_1,x_2) & \sigma_{12}(x_1,x_2) & 0 \\ \sigma_{21}(x_1,x_2) & \sigma_{22}(x_1,x_2) & 0 \\ 0 & 0 & \sigma_{33}(x_1,x_2) \end{bmatrix} \qquad \overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_{11}(x_1,x_2) & \varepsilon_{11}(x_1,x_2) & 0 \\ \varepsilon_{21}(x_1,x_2) & \varepsilon_{22}(x_1,x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Picture: Campus numérique Mecagora



Elasticity problem

Two-dimensional linear elasticity with plane strain : Airy function

$$\varepsilon_{33} = \frac{1+\nu}{E}\sigma_{33} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = 0 \quad \Rightarrow \quad \sigma_{33} = \nu(\sigma_{11}+\sigma_{22})$$

$$\begin{cases} \sigma_{11,1}+\sigma_{12,2}=0 \\ \sigma_{21,1}+\sigma_{22,2}=0 \end{cases} \Rightarrow \begin{cases} \exists\phi(x_1,x_2) \;/\; \sigma_{11}=\phi_{,2} \quad \text{et} \quad \sigma_{12}=-\phi_{,1} \\ \exists\psi(x_1,x_2) \;/\; \sigma_{21}=\psi_{,2} \quad \text{et} \quad \sigma_{22}=-\psi_{,1} \end{cases} \Rightarrow \exists\chi(x_1,x_2) \;/\; \phi=\chi_{,2} \quad \text{et} \quad \psi=-\chi_{,1}$$

$$\exists\chi(x_1,x_2) \;/\; \begin{cases} \sigma_{11}=\chi_{,22} \\ \sigma_{22}=\chi_{,11} \\ \sigma_{12}=-\chi_{,12} \end{cases}$$

Beltrami $\Rightarrow \Delta\Delta\chi = 0$ Airy function

Elasticity ...

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

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Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Thermoelasticity

First law of thermodynamics

First law : energy conservation

$$\frac{d}{dt}(E+K)=P_{ext}+Q$$

Internal energy

$$E=\int_\Omega \rho e\, dv$$

kinetic energy

$$K=\int_\Omega \frac{1}{2}\rho \vec{v}.\vec{v}\, dv$$

Power of external efforts

$$P_{ext}=\int_\Omega \vec{f}.\vec{v}\, dv+\int_{\partial\Omega}\vec{F}.\vec{v}\, ds$$

Heat rate

$$Q=\int_\Omega r\, dv-\int_{\partial\Omega}\vec{q}.\vec{n}\, ds$$

$$\frac{d}{dt}\int_\Omega \rho e\, d\Omega+\frac{d}{dt}\int_\Omega \frac{1}{2}\rho\vec{v}\cdot\vec{v}\, d\Omega=\int_\Omega \vec{f}\cdot\vec{v}\, d\Omega+\int_{\partial\Omega}\vec{F}\cdot\vec{v}\, d\Omega+\int_\Omega r\, d\Omega-\int_{\partial\Omega}\vec{q}\cdot\vec{n}\, d\Omega$$

$$\int_\Omega \rho\frac{de}{dt}\, d\Omega+\int_\Omega \rho\vec{v}\cdot\vec{\gamma}\, d\Omega=\int_\Omega \vec{f}\cdot\vec{v}\, d\Omega+\int_{\partial\Omega}\vec{F}\cdot\vec{v}\, d\Omega+\int_\Omega r\, d\Omega-\int_{\partial\Omega}\vec{q}\cdot\vec{n}\, d\Omega$$

Time derivative of volume integral

$$\int_\Omega \rho\frac{de}{dt}\, d\Omega+\int_\Omega \vec{v}\cdot \mathrm{div}\,\overline{\overline{\sigma}}\, d\Omega=\int_{\partial\Omega}\vec{F}\cdot\vec{v}\, d\Omega+\int_\Omega r\, d\Omega-\int_{\partial\Omega}\vec{q}\cdot\vec{n}\, d\Omega$$

Equilibrium

$$\int_\Omega \rho\frac{de}{dt}\, d\Omega+\int_\Omega \mathrm{div}\left(\overline{\overline{\sigma}}\vec{v}\right) d\Omega-\int_\Omega \overline{\overline{\sigma}}^T:\nabla\vec{v}\, d\Omega=\int_{\partial\Omega}\vec{F}\cdot\vec{v}\, d\Omega+\int_\Omega r\, d\Omega-\int_{\partial\Omega}\vec{q}\cdot\vec{n}\, d\Omega$$

$$\mathrm{Div}\left(\overline{\overline{A}}\vec{u}\right)=\mathrm{div}\,\overline{\overline{A}}^T\cdot\vec{u}+\nabla\vec{u}:\overline{\overline{A}}$$

$$\int_\Omega \rho\frac{de}{dt}\, d\Omega+\int_{\partial\Omega}\left(\overline{\overline{\sigma}}\vec{n}-\vec{F}\right)\cdot\vec{v}\, d\Omega-\int_\Omega \overline{\overline{\sigma}}:\nabla_s\vec{v}\, d\Omega=\int_\Omega r\, d\Omega-\int_\Omega \mathrm{div}\vec{q}\, d\Omega$$

Equilibrium, symetry of the stress tensor and Green theorem

$$\int_\Omega \rho\frac{de}{dt}\, d\Omega-\int_\Omega \overline{\overline{\sigma}}:\overline{\overline{D}}\, d\Omega=\int_\Omega r\, d\Omega-\int_\Omega \mathrm{div}\vec{q}\, d\Omega$$

$$\rho\frac{de}{dt}-\overline{\overline{\sigma}}:\overline{\overline{D}}=r-\mathrm{div}\vec{q}$$

Local formulation of the first law

Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Thermoelasticity

Heat equation

$$\rho\frac{de}{dt} - \overline{\overline{\sigma}} : \overline{\overline{D}} = r - \text{div}\vec{q}$$

In the absence of chemical transformation, in most of the cases, we consider that e=CT, where C is the heat capacity.

According to the Fourier law, the heat flow is proportional to the thermal gradient

$$\vec{q} = -\overline{\overline{k}}\nabla T$$

$\overline{\overline{k}}$ symetric tensor of heat conductivity

$$\rho C\frac{dT}{dt} = \rho C\left(\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T\right) = \overline{\overline{\sigma}} : \overline{\overline{D}} + \text{div}\left(\overline{\overline{k}}\nabla T\right) + r$$

Heat equation

often, the contribution of mechanical origin can be neglected

$$\rho C\frac{dT}{dt} = \text{div}\left(\overline{\overline{k}}\nabla T\right) + r$$

In the stationary case, without heat source and with constant conductivity

$$\Delta T = 0$$

Thermoelasticity

Thermoelasticity

We express the specific internal energy e as a function of the specific entropy s, the temperatureT and the free energy ψ.

$$e = \psi + Ts$$

Thermoelastic assumption: Free energy depends only on temperature and deformations, we only consider small disturbances around an equilibrium state.

$$\rho\psi(\overline{\overline{\varepsilon}},T) = \overline{\overline{\sigma}}_0 : \overline{\overline{\varepsilon}} + \frac{1}{2}\overline{\overline{\varepsilon}} : \overline{\overline{\overline{\overline{C}}}} : \overline{\overline{\varepsilon}} - \rho s\,\delta T - \frac{1}{2}\rho b\,\delta T^2 - \overline{\overline{\beta}} : \overline{\overline{\varepsilon}}\,\delta T$$

$$\overline{\overline{\sigma}} = \rho\frac{\partial\psi}{\partial\overline{\overline{\varepsilon}}}(\overline{\overline{\varepsilon}},T) = \overline{\overline{\sigma}}_0 + \overline{\overline{\overline{\overline{C}}}} : \overline{\overline{\varepsilon}} - \overline{\overline{\beta}}\tau = \overline{\overline{\sigma}}_0 + \overline{\overline{\overline{\overline{C}}}} : \left(\overline{\overline{\varepsilon}} - \overline{\overline{\alpha}}\tau\right)$$

(can be proved starting from Clausius-Duhem inequality)

$$\overline{\overline{\sigma}} = \overline{\overline{\overline{\overline{C}}}} : \left(\overline{\overline{\varepsilon}} - \overline{\overline{\alpha}}\,\delta T\right)$$

General case with zero initial stress

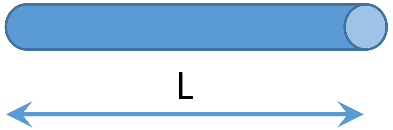
Isotropic case

$$\overline{\overline{\sigma}} = \lambda Tr(\overline{\overline{\varepsilon}})\overline{\overline{I}} + 2\mu\overline{\overline{\varepsilon}} - (3\lambda + 2\mu)\alpha\delta T\overline{\overline{I}} \qquad \overline{\overline{\varepsilon}} = \alpha\delta T\overline{\overline{I}} - \frac{\nu}{E}Tr\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}} + \frac{1+\nu}{E}\overline{\overline{\sigma}}$$

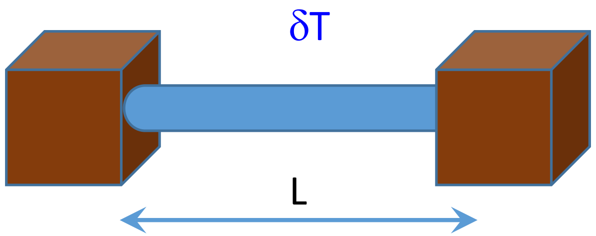
Thermoelasticity

Illustration

« Thermal » strain



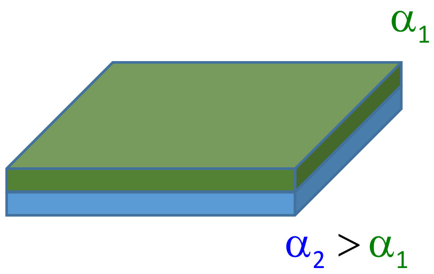
$$\sigma_{11} = 0 \qquad \varepsilon_{11} = \frac{\partial u_1}{\partial X_1} = \alpha \delta T \qquad u_1(L) = \alpha \delta T L$$

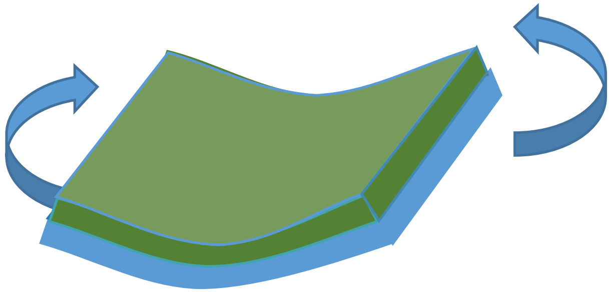


$$\varepsilon_{11} = 0 \qquad \sigma_{11} = -E\alpha\delta T$$

« thermal » stress

$\delta T > 0$





« thermal » strain and stress

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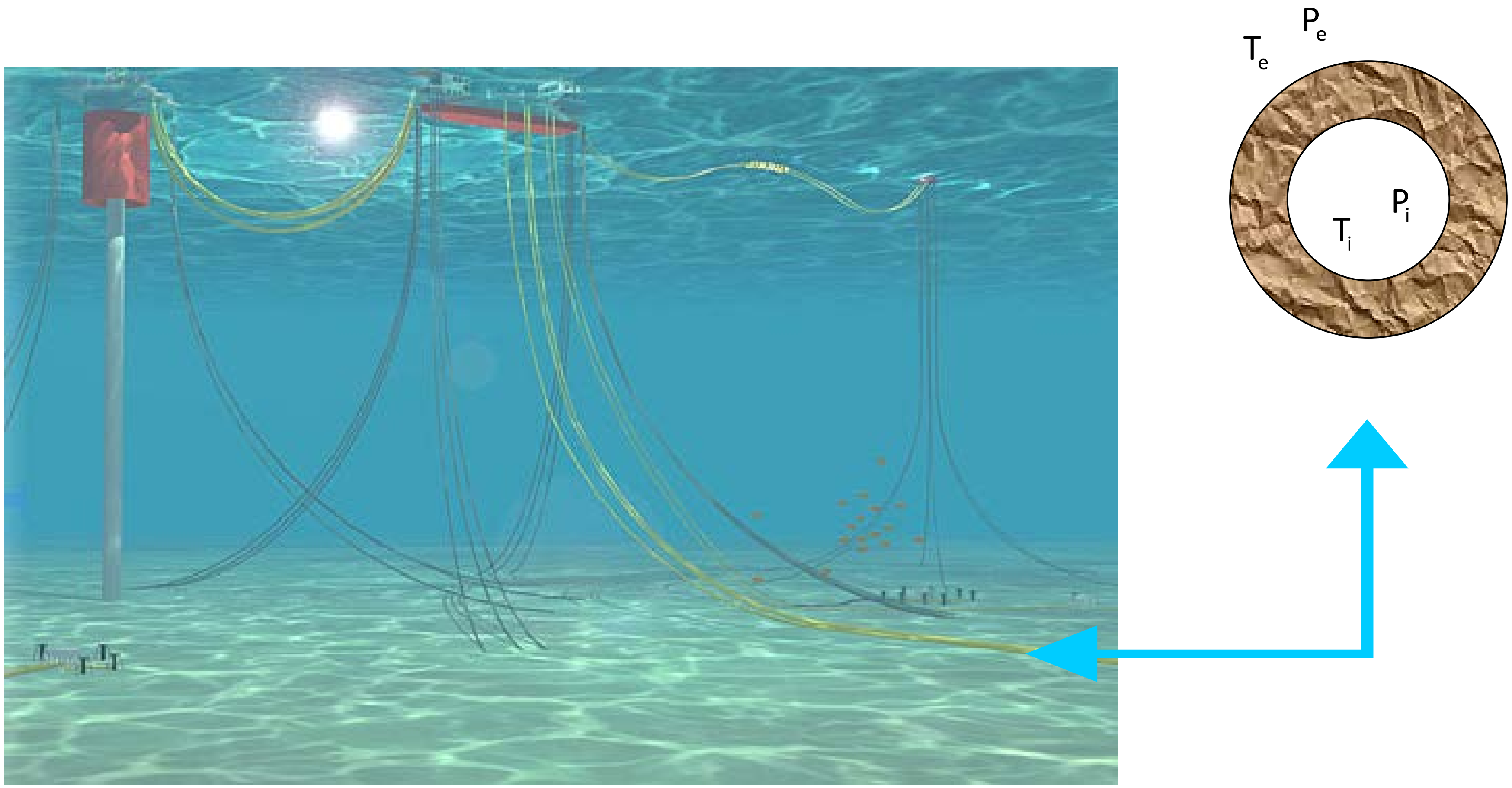
Example



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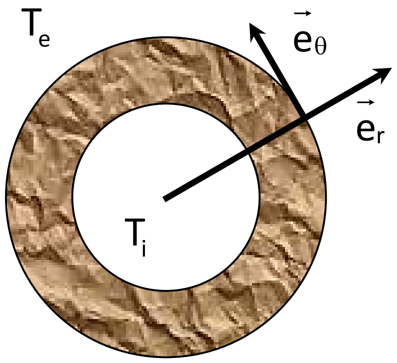
Resolution



Thermal problem

$$r + \operatorname{div}\left(\bar{\bar{k}}\nabla T\right) = \rho C \frac{dT}{dt} - T\frac{\partial \bar{\bar{\sigma}}}{\partial T} : \frac{\partial \bar{\bar{\varepsilon}}}{\partial t} \rightarrow \Delta T = 0$$

$$T = T(r) \rightarrow \delta T = T(r) - T_0 = a\ln r + b$$

with $T(r_i) = T_i$ et $T(r_e) = T_e$

Mechanical problem

Assumption $\vec{u} = u(r)\vec{e}_r$

so $$\bar{\bar{\varepsilon}} = \begin{bmatrix} u' & 0 & 0 \\ 0 & u/r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ and

$$\sigma_{rr} = \lambda\left(u' + \frac{u}{r}\right) + 2\mu u' - (3\lambda + 2\mu)\alpha\delta T(r)$$

$$\sigma_{\theta\theta} = \lambda\left(u' + \frac{u}{r}\right) + 2\mu\frac{u}{r} - (3\lambda + 2\mu)\alpha\delta T(r)$$

$$\operatorname{div}\bar{\bar{\sigma}} + \vec{f} = \rho\vec{\gamma} \rightarrow \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \rightarrow u'' + \frac{u'}{r} - \frac{u}{r^2} = \frac{(3\lambda + 2\mu)a\alpha}{(\lambda + 2\mu)r}$$

$$u(r) = \frac{(3\lambda + 2\mu)a\alpha}{(\lambda + 2\mu)} r\,Ln(r) + Ar + \frac{B}{r}$$

with $\sigma_{rr}(r_e) = -P_e$ et $\sigma_{rr}(r_i) = -P_i$

Elasticity …

Constitutive law

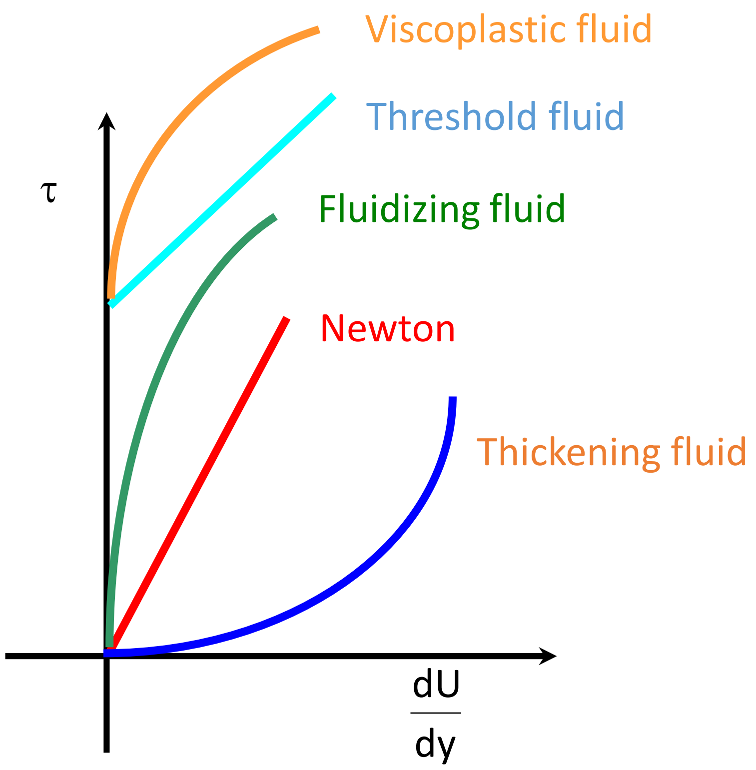
Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Introduction to fluid mechanics

Constitutive law



Newtonian fluid

$$\overline{\overline{\sigma}} = -p\overline{\overline{1}} + \lambda tr\left(\dot{\overline{\overline{\varepsilon}}}\right)\overline{\overline{1}} + 2\mu\dot{\overline{\overline{\varepsilon}}}$$

$$\rho\vec{\gamma} = div\overline{\overline{\sigma}} + \vec{f}$$

$$\rho\frac{d\vec{v}}{dt} = div\left(-p\overline{\overline{1}} + \lambda tr\left(\dot{\overline{\overline{\varepsilon}}}\right)\overline{\overline{1}} + 2\mu\dot{\overline{\overline{\varepsilon}}}\right)$$

$$\rho\frac{d\vec{v}}{dt} = -div\left(p\overline{\overline{1}}\right)\lambda div\left(div\vec{v}\overline{\overline{1}}\right) + \mu div\left(\nabla\vec{v} + \nabla^T\vec{v}\right)$$

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + \nabla\vec{v}.\vec{v}\right) = -\nabla p + \lambda\nabla\left(div\vec{v}\right) + \mu div\left(\nabla\vec{v}\right) + \mu div\left(\nabla^T\vec{v}\right)$$

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho\nabla\vec{v}.\vec{v} = -\nabla p + (\lambda + \mu)\nabla\left(div\vec{v}\right) + \mu div\left(\nabla\vec{v}\right)$$

If the fluid is incompressible, then $div\vec{v} = 0$

$$\frac{\partial\vec{v}}{\partial t} + \nabla\vec{v}.\vec{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\vec{v}$$

Incompressible Navier-Stokes equations

$$\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=0$$

$$\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}+v_3\frac{\partial v_1}{\partial x_3}=-\frac{1}{\rho}\frac{\partial p}{\partial x_1}+\nu\left(\frac{\partial^2 v_1}{\partial x_1^2}+\frac{\partial^2 v_1}{\partial x_2^2}+\frac{\partial^2 v_1}{\partial x_3^2}\right)$$

$$\frac{\partial v_2}{\partial t}+v_1\frac{\partial v_2}{\partial x_1}+v_2\frac{\partial v_2}{\partial x_2}+v_3\frac{\partial v_2}{\partial x_3}=-\frac{1}{\rho}\frac{\partial p}{\partial x_2}+\nu\left(\frac{\partial^2 v_2}{\partial x_1^2}+\frac{\partial^2 v_2}{\partial x_2^2}+\frac{\partial^2 v_2}{\partial x_3^2}\right)$$

$$\frac{\partial v_3}{\partial t}+v_1\frac{\partial v_3}{\partial x_1}+v_2\frac{\partial v_3}{\partial x_2}+v_3\frac{\partial v_3}{\partial x_3}=-\frac{1}{\rho}\frac{\partial p}{\partial x_3}+\nu\left(\frac{\partial^2 v_3}{\partial x_1^2}+\frac{\partial^2 v_3}{\partial x_2^2}+\frac{\partial^2 v_3}{\partial x_3^2}\right)$$

Elasticity …

Constitutive law

Elasticity problem

Thermoelasticity

Introduction to fluid mechanics

Introduction to fluid mechanics

Example: plane Couette-Poiseuille flow

We consider a stationary flow between two infinite plates, one of which is motionless while the other is animated with a constant velocity U

$$\frac{\partial}{\partial t}=0\,,\quad \frac{\partial}{\partial x_3}=0\,,\quad v_2=v_3=0$$

$$\frac{\partial v_1}{\partial x_1}=0$$

$$v_1\frac{\partial v_1}{\partial x_1}=-\frac{1}{\rho}\frac{\partial p}{\partial x_1}+\nu\left(\frac{\partial^2 v_1}{\partial x_1^2}+\frac{\partial^2 v_1}{\partial x_2^2}\right)$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x_2}$$

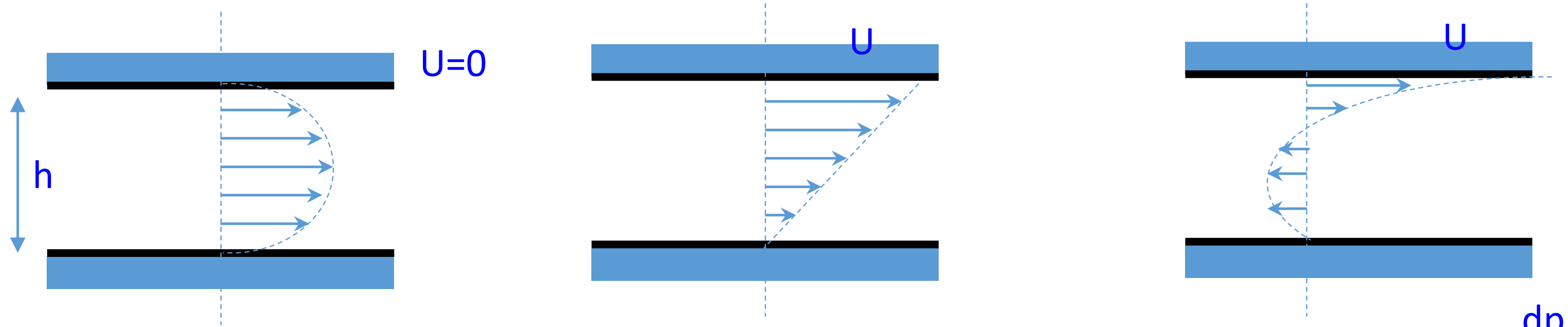
$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x_3}$$

$$p=p(x_1)\,,\,v_1=v_1(x_2)$$

$$\nu\frac{\partial^2 v_1}{\partial x_2^2}=\frac{1}{\rho}\frac{dp}{dx_1}$$

$$p=p(x_1)\,,\,v_1=v_1(x_2)$$

$$v_1(x_2)=\mu\frac{dp}{dx_1}x_2(x_2-h)+U\frac{x_2}{h}$$



Case U=0 : 2d Poiseuille, parabolic

Case p=cste : 2d Couette linear shape

Mixed case with $\frac{dp}{dx_1}>0$